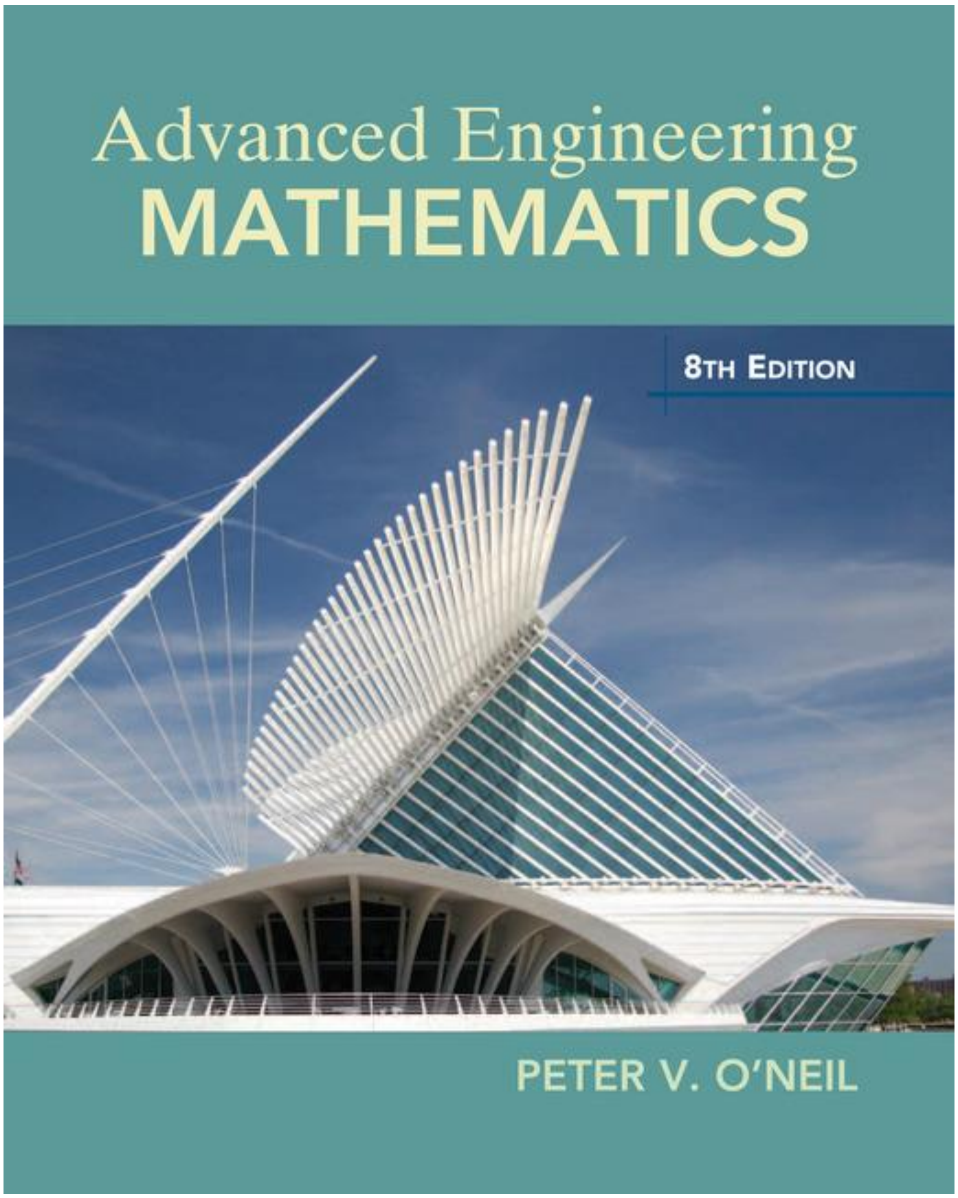
A Student's Solutions Manual to Accompany

ADVANCED ENGINEERING MATHEMATICS, 8TH EDITION

PETER V. O'NEIL



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STUDENT'S SOLUTIONS MANUAL

TO ACCOMPANY

Advanced Engineering Mathematics

8th EDITION

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Contents

Chapter 1

First-Order Differential Equations

1.1 Terminology and Separable Equations

1. The differential equation is separable because it can be written

$$3y^2 \frac{dy}{dx} = 4x,$$

or, in differential form,

$$3y^2 \, dy = 4x \, dx.$$

Integrate to obtain

$$y^3 = 2x^2 + k.$$

This implicitly defines a general solution, which can be written explicitly as

$$y = (2x^2 + k)^{1/3},$$

with k an arbitrary constant.

3. If $\cos(y) \neq 0$, the differential equation is

$$\begin{aligned} \frac{y}{dx} &= \frac{\sin(x+y)}{\cos(y)} \\ &= \frac{\sin(x)\cos(y) + \cos(x)\sin(y)}{\cos(y)} \\ &= \sin(x) + \cos(x)\tan(y). \end{aligned}$$

There is no way to separate the variables in this equation, so the differential equation is not separable.

5. The differential equation can be written

$$x\frac{dy}{dx} = y^2 - y,$$

or

$$\frac{1}{y(y-1)}\,dy = \frac{1}{x}\,dx,$$

and is therefore separable. Separating the variables assumes that $y \neq 0$ and $y \neq 1$. We can further write

$$\left(\frac{1}{y-1} - \frac{1}{y}\right)\,dy = \frac{1}{x}\,dx.$$

Integrate to obtain

$$\ln|y-1| - \ln|y| = \ln|x| + k.$$

Using properties of the logarithm, this is

$$\ln\left|\frac{y-1}{xy}\right| = k.$$

Then

$$\frac{y-1}{xy} = c,$$

with $c = e^k$ constant. Solve this for y to obtain the general solution

$$y = \frac{1}{1-cx}.$$

$y = 0$ and $y = 1$ are singular solutions because these satisfy the differential equation, but were excluded in the algebra of separating the variables.

7. The equation is separable because it can be written in differential form as

$$\frac{\sin(y)}{\cos(y)}\,dy = \frac{1}{x}\,dx.$$

This assumes that $x \neq 0$ and $\cos(y) \neq 0$. Integrate this equation to obtain

$$-\ln|\cos(y)| = \ln|x| + k.$$

This implicitly defines a general solution. From this we can also write

$$\sec(y) = cx$$

with c constant.

The algebra of separating the variables required that $\cos(y) \neq 0$. Now $\cos(y) = 0$ if $y = (2n+1)\pi/2$, with n any integer. Now $y = (2n+1)\pi/2$ also satisfies the original differential equation, so these are singular solutions.

9. The differential equation is

$$\frac{dy}{dx} = e^x - y + \sin(y),$$

and this is not separable. It is not possible to separate all terms involving x on one side of the equation and all terms involving y on the other.

11. If $y \neq -1$ and $x \neq 0$, we obtain the separated equation

$$\frac{y^2}{y+1}\,dy = \frac{1}{x}\,dx.$$

To make the integration easier, write this as

$$\left(y - 1 + \frac{1}{1+y}\right)\,dy = \frac{1}{x}\,dx.$$

Integrate to obtain

$$\frac{1}{2}y^2 - y + \ln|1+y| = \ln|x| + c.$$

This implicitly defines a general solution. The initial condition is $y(3e^2) = 2$, so put $y = 2$ and $x = 3e^2$ to obtain

$$2 - 2 + \ln(3) = \ln(3e^2) + c.$$

Now

$$\ln(3e^2) = \ln(3) + \ln(e^2) = \ln(3) + 2,$$

so

$$\ln(3) = \ln(3) + 2 + c.$$

Then $c = -2$ and the solution of the initial value problem is implicitly defined by

$$\frac{1}{2}y^2 - y + \ln|1+y| = \ln|x| - 2.$$

13. With $\ln(y^x) = x\ln(y)$, we obtain the separated equation

$$\frac{\ln(y)}{y}\,dy = 3x\,dx.$$

Integrate to obtain

$$(\ln(y))^2 = 3x^2 + c.$$

For $y(2) = e^3$, we need

$$(\ln(e^3))^2 = 3(4) + c,$$

or $9 = 12 + c$. Then $c = -3$ and the solution of the initial value problem is defined by

$$(\ln(y))^2 = 3x^2 - 3.$$

Solve this to obtain the explicit solution

$$y = e^{\sqrt{3(x^2-1)}}$$

if $|x| > 1$.

15. Separate the variables to obtain

$$y\cos(3y)\,dy = 2x\,dx.$$

Integrate to get

$$\frac{1}{3}y\sin(3y) + \frac{1}{9}\cos(3y) = x^2 + c,$$

which implicitly defines a general solution. For $y(2/3) = \pi/3$, we need

$$\frac{1}{3}\frac{\pi}{3}\sin(\pi) + \frac{1}{9}\cos(\pi) = \frac{4}{9} + c.$$

This reduces to

$$-\frac{1}{9} = \frac{4}{9} + c,$$

so $c = -5/9$ and the solution of the initial value problem is implicitly defined by

$$\frac{1}{3}y\sin(3y) + \frac{1}{9}\cos(3y) = x^2 - \frac{5}{9},$$

or

$$3y\sin(3y) + \cos(3y) = 9x^2 - 1.$$

17. Suppose the thermometer was removed from the house at time $t = 0$, and let $T(t)$ be the temperature function. Let A be the ambient temperature outside the house (assumed constant). By Newton's law,

$$T'(t) = k(t - A).$$

We are also given that $T(0) = 70$ and $T(5) = 60$. Further, fifteen minutes after being removed from the house, the thermometer reads 50.4, so

$$T(15) = 50.4.$$

We want to determine A, the constant outside temperature. From the differential equation for T,

$$\frac{1}{T - A}\,dT = kdt.$$

Integrate this, as we have done before, to get

$$T(t) = A + ce^{kt}.$$

Now,

$$T(0) = 70 = A + c,$$

so $c = 70 - A$ and

$$T(t) = A + (70 - A)e^{kt}.$$

Now use the other two conditions:

$$T(5) = A + (70 - A)e^{5k} = 15.5 \text{ and } T(15) = A + (70 - A)e^{15k} = 50.4.$$

From the equation for $T(5)$, solve for e^{5k} to get

$$e^{5k} = \frac{60 - A}{70 - A}.$$

Then

$$e^{15k} = \left(e^{5k}\right)^3 = \left(\frac{60 - A}{70 - A}\right)^3.$$

Substitute this into the equation $T(15)$ to get

$$(70 - A)\left(\frac{60 - A}{70 - A}\right)^3 = 50.4 - A.$$

Then

$$(60 - A)^3 = (50.4 - A)(70 - A)^2.$$

The cubic terms cancel and this reduces to the quadratic equation

$$10.4A^2 - 1156A + 30960 = 0,$$

with roots 45 and (approximately) 66.15385. Clearly the outside temperature must be less than 50, and must therefore equal 45 degree.

19. The problem is like Problem 18, and we find that the amount of Uranium-235 at time t is

$$U(t) = 10\left(\frac{1}{2}\right)^{t/(4.5(10^9))},$$

with t in years. Then

$$U(10^9) = 10\left(\frac{1}{2}\right)^{1/4.5} \approx 8.57 \text{ kg.}$$

21. Let

$$I(x) = \int_0^\infty e^{-t^2-(x/t)^2}\,dt.$$

The integral we want is $I(3)$. Compute

$$I'(x) = -2x\int_0^\infty \frac{1}{t^2}e^{-t^2-(x/t)^2}\,dt.$$

Let $u = x/t$, so $t = x/u$ and

$$dt = -\frac{x}{u^2}\,du.$$

Then

$$\begin{aligned} I'(x) &= -2x \int_\infty^0 \left(\frac{u^2}{x^2}\right) e^{-(x/u)^2-u^2} \frac{-x}{u^2}\, du \\ &= -2I(x). \end{aligned}$$

Then $I(x)$ satisfies the separable differential equation $I' = -2I$, with general solution of the form $I(x) = ce^{-2x}$. Now observe that

$$I(0) = \int_0^\infty e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2} = c,$$

in which we used a standard integral that arises often in statistics. Then

$$I(x) = \frac{\sqrt{\pi}}{2} e^{-2x}.$$

Finally, put $x = 3$ for the particular integral of interest:

$$I(3) = \int_0^\infty e^{-t^2-(9/t)^2}\, dt = \frac{\sqrt{\pi}}{2} e^{-6}.$$

23. With a and b as given, and $p_0 = 3,929,214$ (the population in 1790), the logistic population function for the United States is

$$P(t) = \frac{123,141.5668}{0.03071576577 + 0.0006242342283e^{0.03134t}} e^{0.03134t}.$$

If we attempt an exponential model $Q(t) = Ae^{kt}$, then take $A = Q(0) = 3,929,214$, the population in 1790. To find k, use the fact that

$$Q(10) = 5308483 = 3929214e^{10k}$$

and we can solve for k to get

$$k = \frac{1}{10} \ln\left(\frac{5308483}{3929214}\right) \approx 0.03008667012.$$

The exponential model, using these two data points (1790 and 1800 populations), is

$$Q(t) = 3929214e^{0.03008667012t}.$$

Table 1.1 uses $Q(t)$ and $P(t)$ to predict later populations from these two initial figures. The logistic model remains quite accurate until about 1960, at which time it loses accuracy quickly. The exponential model becomes quite inaccurate by 1870, after which the error becomes so large that it is not worth computing further. Exponential models do not work well over time with complex populations, such as fish in the ocean or countries throughout the world.

year	population	$P(t)$	percent error	$Q(t)$	percent error
1790	3,929,213	3,929,214	0	3,929,214	0
1800	5,308,483	5,336,313	0.52	5,308,483	0
1810	7,239,881	7,228,171	-0.16	7,179,158	-0.94
1820	9,638,453	9,757,448	1.23	7,179,158	0.53
1830	12,886,020	13,110,174	1.90	13,000,754	1.75
1840	17,169,453	17,507,365	2.57	17,685,992	3.61
1850	23,191,876	23.193,639	0.008	23,894,292	3.03
1860	31,443,321	30,414,301	-3.27	32,281,888	2.67
1870	38,558,371	39,374,437	2.12	43,613,774	13.11
1880	50,189,209	50,180,383	-0.018	58,923,484	17.40
1890	62,979766	62,772,907	-0.33	79,073,491	26.40
1900	76,212,168	76,873,907	0.87	107,551,857	41.12
1910	92,228,496	91,976,297	-0.27	145,303,703	57.55
1920	106,021,537	107,398,941	1.30	196,312,254	83.16
1930	123,202,624	122,401,360	-0.65		
1940	132,164,569	136,329,577	3.15		
1950	151,325,798	148,679,224	-1.75		
1960	179,323,175	150,231,097	-11.2		
1970	203,302,031	167,943,428	-17.39		
1980	226,547,042	174,940,040	-22.78		

Table 1.1: Census data for Problem 23

1.2 The Linear First-Order Equation

1. With $p(x) = -3/x$, and integrating factor is

$$e^{\int(-3/x)\,dx} = e^{-3\ln(x)} = x^{-3}$$

for $x > 0$. Multiply the differential equation by x^{-3} to get

$$x^{-3}y' - 3x^{-4} = 2x^{-1}.$$

or

$$\frac{d}{dx}(x^{-3}y) = \frac{2}{x}.$$

Integrate to get

$$x^{-3}y = 2\ln(x) + c,$$

with c an arbitrary constant. For $x > 0$ we have a general solution

$$y = 2x^3\ln(x) + cx^3.$$

In the last integration, we can allow $x < 0$ by replacing $\ln(x)$ with $\ln|x|$ to derive the solution

$$y = 2x^3\ln|x| + cx^3$$

for $x \neq 0$.

3. $e^{\int 2\,dx} = e^{2x}$ is an integrating factor. Multiply the differential equation by e^{2x}:

$$y'e^{2x} + 2ye^{2x} = xe^{2x},$$

or

$$(e^{2x}y)' = xe^{2x}.$$

Integrate to get

$$e^{2x}y = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c.$$

giving us the general solution

$$y = \frac{1}{2}x - \frac{1}{4} + ce^{-2x}.$$

5. First determine the integrating factor

$$e^{\int -2\,dx} = e^{-2x}.$$

Multiply the differential equation by e^{-2x} to get

$$(e^{-2x}y)' = -8x^2e^{-2x}.$$

Integrate to get

$$e^{-2x}y = \int -8x^2e^{-2x}\,dx = 4x^2e^{-2x} + 4xe^{-2x} + 2e^{-2x} + c.$$

This yields the general solution

$$y = 4x^2 + 4x + 2 + ce^{2x}.$$

7. $x-2$ is an integrating factor for the differential equation because

$$e^{\int (1/(x-2))\,dx} = e^{\ln(x-2)} = x-2.$$

Multiply the differential equation by $x-2$ to get

$$((x-2)y)' = 3x(x-2).$$

Integrate to get

$$(x-2)y = x^3 - 3x^2 + c.$$

This gives us the general solution

$$y = \frac{1}{x-2}(x^3 - 3x^2 + c).$$

Now we need

$$y(3) = 27 - 27 + c = 4,$$

so $c = 4$ and the solution of the initial value problem is

$$y = \frac{1}{x-2}(x^3 - 3x^2 + 4).$$

9. First derive the integrating factor

$$e^{\int (2/(x+1))\,dx} = e^{2\ln(x+1)} = e^{\ln((x+1)^2)} = (x+1)^2.$$

Multiply the differential equation by $(x+1)^2$ to obtain

$$\left((x+1)^2 y\right)' = 3(x+1)^2.$$

Integrate to obtain

$$(x+1)^2 y = (x+1)^3 + c.$$

Then

$$y = x + 1 + \frac{c}{(x+1)^2}.$$

Now

$$y(0) = 1 + c = 5$$

so $c = 4$ and the initial value problem has the solution

$$y = x + 1 + \frac{4}{(x+1)^2}.$$

11. Let (x, y) be a point on the curve. The tangent line at (x, y) must pass through $(0, 2x^2)$, and so has slope

$$y' = \frac{y - 2x^2}{x}.$$

This is the linear differential equation

$$y' - \frac{1}{x}y = -2x.$$

An integrating factor is

$$e^{-\int (1/x)\,dx} = e^{-\ln(x)} = e^{\ln(1/x)} = \frac{1}{x},$$

so multiply the differential equation by $1/x$ to get

$$\frac{1}{x}y' - \frac{1}{x^2}y = -2.$$

This is

$$\left(\frac{1}{x}y\right)' = -2.$$

Integrate to get

$$\frac{1}{x}y = -2x + c.$$

Then

$$y = -2x^2 + cx,$$

in which c can be any number.

13. Let $A_1(t)$ and $A_2(t)$ be the number of pounds of salt in tanks 1 and 2, respectively, at time t. Then

$$A_1'(t) = \frac{5}{2} - \frac{5A_1(t)}{100}; A_1(0) = 20$$

and

$$A_2'(t) = \frac{5A_1(t)}{100} - \frac{5A_2(t)}{150}; A_2(0) = 90.$$

Solve the linear initial value problem for $A_1(t)$ to get

$$A_1(t) = 50 - 30e^{-t/20}.$$

Substitute this into the differential equation for $A_2(t)$ to get

$$A_2' + \frac{1}{30}A_2 = \frac{5}{2} - \frac{3}{2}e^{-t/20}; A_2(0) = 90.$$

Solve this linear problem to obtain

$$A_2(t) = 75 + 90e^{-t/20} - 75e^{-t/30}.$$

Tank 2 has its minimum when $A_2'(t) = 0$, and this occurs when

$$2.5e^{-t/30} - 4.5e^{-t/20} = 0.$$

This occurs when $e^{t/60} = 9/5$, or $t = 60\ln(9/5)$. Then

$$A_2(t)_{\min} = A_2(60\ln(9/5)) = \frac{5450}{81}$$

pounds.

1.3 Exact Equations

In these problems it is assumed that the differential equation has the form $M(x,y)+N(x,y)y'=0$, or, in differential form, $M(x,y)\,dx+N(x,y)\,dy=0$.

1. With $M(x,y)=2y^2+ye^{xy}$ and $N(x,y)=4xy+xe^{xy}+2y$. Then

$$\frac{\partial N}{\partial x}=4y+e^{xy}+xye^{xy}=\frac{\partial M}{\partial y}$$

for all (x,y), so the differential equation is exact on the entire plane. A potential function $\varphi(x,y)$ must satisfy

$$\frac{\partial \varphi}{\partial x}=M(x,y)=2y^2+ye^{xy}$$

and

$$\frac{\partial \varphi}{\partial y}=N(x,y)=4xy+xe^{xy}+2y.$$

Choose one to integrate. If we begin with $\partial\varphi/\partial x=M$, then integrate with respect to x to get

$$\varphi(x,y)=2xy^2+e^{xy}+\alpha(y),$$

with $\alpha(y)$ the "constant" of integration with respect to x. Then we must have

$$\frac{\partial \varphi}{\partial y}=4xy+xe^{xy}+\alpha'(y)=4xy+xe^{xy}+2y.$$

This requires that $\alpha'(y)=2y$, so we can choose $\alpha(y)=y^2$ to obtain the potential function

$$\varphi(x,y)=2xy^2+e^{xy}+y^2.$$

The general solution is defined implicitly by the equation

$$2xy^2+e^{xy}+y^2=c,,$$

with c an arbitrary constant.

3. $\partial M/\partial y=4x+2x^2$ and $\partial N/\partial x=4x$, so this equation is not exact (on any rectangle).

5. $\partial M/\partial y=1=\partial N/\partial x$, for $x\neq 0$, so this equation is exact on the plane except at points $(0,y)$. Integrate $\partial\varphi/\partial x=M$ or $\partial\varphi/\partial y=N$ to find the potential function

$$\varphi(x,y)=\ln|x|+xy+y^3$$

for $x\neq 0$. The general solution is defined by an equation

$$\ln|x|+xy+y^3=k.$$

7. For this equation to be exact, we need

$$\frac{\partial M}{\partial y} = 6xy^2 - 3 = \frac{\partial N}{\partial x} = -3 - 2\alpha xy^2.$$

This will be true if $\alpha = -3$. By integrating, we find a potential function

$$\varphi(x, y) = x^2y^3 - 3xy - 3y^2$$

and a general solution is defined implicitly by

$$x^2y^3 - 3xy - 3y^2 = k.$$

9. Because $\partial M/\partial y = 12y^2 = \partial N/\partial x$, this equation is exact for all (x, y). Straightforward integrations yield the potential function

$$\varphi(x, y) = 3xy^4 - x.$$

A general solution is defined implicitly by

$$3xy^4 - x = k.$$

To satisfy the condition $y(1) = 2$, we must choose k so that

$$48 - 1 = k,$$

so $k = 47$ and the solution of the initial value problem is specified by the equation

$$3xy^4 - x = 47.$$

In this case we can actually write this solution explicitly with y in terms of x.

11. First,

$$\frac{\partial M}{\partial y} = -2x\sin(2y - x) - 2\cos(2y - x) = \frac{\partial N}{\partial x},$$

so the differential equation is exact for all (x, y). For a potential function, integrate

$$\frac{\partial \varphi}{\partial y} = -2x\cos(2y - x)$$

with respect to y to get

$$\varphi(x, y) = -x\sin(2y - x) + c(x).$$

Then we must have

$$\begin{aligned}\frac{\partial \varphi}{\partial x} &= x\cos(2y - x) - \sin(2y - x)\\ &= x\cos(2y - x) - \sin(2y - x) + c'(x).\end{aligned}$$

Then $c'(x) = 0$ and we can take $c(x)$ to be any constant. Choosing $c(x) = 0$ yields

$$\varphi(x, y) = -x\sin(2y - x).$$

The general solution is defined implicitly by

$$-x\sin(2y - x) = k.$$

To satisfy $y(\pi/12) = \pi/8$, we need

$$-\frac{\pi}{12}\sin(\pi/6) = k,$$

so choose $k = -\pi/24$ to obtain the solution defined by

$$-x\sin(2y - x) = -\frac{\pi}{24}$$

which of course is the same as

$$x\sin(2y - x) = \frac{\pi}{24}.$$

We can also write

$$y = \frac{1}{2}\left(x + \arcsin\left(\frac{\pi}{24x}\right)\right)$$

for $x \neq 0$.

13. $\varphi + c$ is also a potential function if φ is because

$$\frac{\partial\varphi}{\partial x} = \frac{\partial(\varphi + c)}{\partial x}$$

and

$$\frac{\partial\varphi}{\partial y} = \frac{\partial(\varphi + c)}{\partial y}.$$

The function defined implicitly by

$$\varphi(x, y) = k$$

is the same as that defined by

$$\varphi(x, y) + c = k$$

if k is arbitrary.

15. First,

$$\frac{\partial M}{\partial y} = x - \frac{3}{2}y^{-5/2} \text{ and } \frac{\partial N}{\partial x} = 2x.$$

and these are not equal on any rectangle in the plane.

In differential form, the differential equation is

$$(xy + y^{-3/2})\,dx + x^2\,dy = 0.$$

Multiply this equation by $x^a y^b$ to get

$$(x^{a+1}y^{b+1} + x^a y^{b-3/2})\,dx + x^{a+2}y^b\,dy = 0 = M^*\,dx + N^*\,dy.$$

For this to be exact, we need

$$\begin{aligned}\frac{\partial M^*}{\partial y} &= (b+1)x^{a+1}y^b + \left((b - \frac{3}{2}\right) x^a y^{b-5/2} \\ &= \frac{\partial N^*}{\partial x} = (a+2)x^{a+1}y^b.\end{aligned}$$

Divide this equation by $x^a y^b$ to get

$$(b+1)x + \left(b - \frac{3}{2}\right) y^{-5/2} = (a+2)x.$$

This will hold for all x and y if we let $b = 3/2$ and then choose a and b so that $b + 1 = a + 2$. Thus choose

$$a = \frac{1}{2} \text{ and } b = \frac{3}{2}$$

to get the integrating factor $\mu(x, y) = x^{1/2}b^{3/2}$. Multiply the original differential equation by this to get

$$(x^{3/2}y^{5/2} + x^{1/2})\,dx + x^{5/2}y^{1/2}\,dy = 0.$$

To find a potential function, integrate

$$\frac{\partial \varphi}{\partial y} = x^{5/2}y^{3/2}$$

with respect to y to get

$$\varphi(x, y) = \frac{2}{5}x^{5/2}y^{5/2} + c(x).$$

Then we need

$$\frac{\partial \varphi}{\partial x} = x^{3/2}y^{5/2} + c'(x) = x^{3/2}y^{5/2} + x^{1/2}.$$

Therefore $c'(x) = x^{1/2}$, so $c(x) = 2x^{3/2}/3$ and

$$\varphi(x) = \frac{2}{5}x^{5/2}y^{5/2} + \frac{2}{3}x^{3/2}.$$

The general solution of the original differential equation is given implicitly by

$$\frac{2}{5}(xy)^{5/2} + \frac{2}{3}x^{3/2} = k.$$

In this we must have $x \neq 0$ and $y \neq 0$ to ensure that the integrating factor $\mu(x, y) \neq 0$.

1.4 Homogeneous, Bernoulli and Riccati Equations

1. This is a Riccati equation and one solution (by inspection) is $S(x) = x$. Let $y = x + 1/z$ to obtain

$$2 - \frac{1}{z^2}z' = \frac{1}{x^2}\left(x + \frac{1}{z}\right)^2 - \frac{1}{x}\left(x + \frac{1}{z}\right) + 1.$$

This simplifies to

$$z' + \frac{1}{x}z = -\frac{1}{x^2},$$

a linear equation with integrating factor

$$e^{\int (1/x)\,dx} = e^{\ln(x)} = x.$$

The differential equation for z can therefore be written

$$(xz)' = -\frac{1}{x}.$$

Integrate to get

$$xz = -\ln(x) + c,$$

so

$$z = -\frac{\ln(x)}{x} + \frac{c}{x} = \frac{c - \ln(x)}{x}.$$

for $x > 0$. Then

$$y = x + \frac{1}{z} = x + \frac{x}{c - \ln(x)}$$

for $x > 0$.

3. This is a Bernoulli equation with $\alpha = 2$, so let $v = y^{1-\alpha} = y^{-1}$ for $y \neq 0$ and $y = 1/v$. Compute

$$y' = \frac{dy}{dv}\frac{dv}{dx} = -\frac{1}{v^2}xv'.$$

The differential equation becomes

$$-\frac{1}{v^2}v' + \frac{x}{v} = \frac{x}{v^2}.$$

This is

$$v' - xv = -x,$$

a linear equation with integrating factor $e^{-x^2/2}$. We can therefore write

$$(e^{-x^2/2}v)' = -xe^{-x^2/2}.$$

Integrate to get

$$e^{-x^2/2}v = e^{-x^2/2} + c,$$

so

$$v = 1 + ce^{-x^2/2}.$$

The original differential equation has the general solution

$$y = \frac{1}{v} = \frac{1}{1 + ce^{-x^2/2}},$$

in which c is an arbitrary constant.

5. This differential equation is homogeneous and setting $y = ux$ gives us

$$u + xu' = \frac{u}{1+u}.$$

This is the separable equation

$$x\frac{du}{dx} = \frac{u}{1+u} - u$$

which, in terms of x and y, is

$$\left(\frac{1}{u^2} + \frac{1}{u}\right) du = -\frac{1}{x}\,dx.$$

Integrate to get

$$\frac{1}{u} + \ln|u| = -\ln|x| + c.$$

With $u = y/x$ this reduces to

$$-x + y\ln|y| = cy,$$

with c an arbitrary constant.

6. This is a Riccati equation and one solution (by inspection) is $S(x) = 4$. After some routine computation we obtain the general solution

$$y = 4 + \frac{6x^3}{c - x^3}.$$

7. The differential equation is exact, with general solution defined implicitly by

$$xy - x^2 - y^2 = c.$$

9. The differential equation is of Bernoulli type with $\alpha = -3/4$. The general solution is defined by

$$5(xy)^{7/4} + 7x^{-5/4} = c.$$

11. The equation is Bernoulli with $\alpha = 2$ and the change of variables $v = y^{-1}$ leads to the general solution

$$y = 2 + \frac{2}{cx^2 - 1}.$$

13. The differential equation is Riccati and one solution is $S(x) = e^x$. A general solution is given explicitly by

$$y = \frac{2e^x}{ce^{2x} - 1}.$$

15. For the first part,

$$F\left(\frac{ax + by + c}{dx + py + r}\right) = F\left(\frac{a + b(y/x)c/x}{d + p(y/x) + r/x}\right) = f\left(\frac{y}{x}\right)$$

if and only if $c = r = 0$.

Next, suppose $x = X + h$ and $y = Y + k$. Then

$$\begin{aligned}\frac{dY}{dX} &= F\left(\frac{a(X + h) + b(Y + k) + c}{d(x + h) + p(Y + k) + r}\right)\\ &= F\left(\frac{aX + bY + c + ah + bk + c}{dX + pY + r + dh + pk + r}\right).\end{aligned}$$

This equation is homogeneous exactly when h and k can be chosen so that

$$ah + bk = -c \text{ and } dh + pk = -r.$$

This 2×2 system of algebraic equations has a solution exactly when the determinant of the coefficients is nonzero, and this is the condition that

$$\begin{vmatrix} a & b \\ d & p \end{vmatrix} = ap - bd \neq 0.$$

17. Let $X = x - 2, Y = y + 3$ to get the homogeneous equation

$$\frac{dY}{dX} = \frac{3X - Y}{X + Y}.$$

The general solution of the original equation (in terms of x and y) is defined by

$$3(x - 2)^2 - 2(x - 2)(y + 3) - (y + 3)^2 = c,$$

with c an arbitrary constant.

19. Let $X = x - 2, Y = y + 1$ to obtain the general solution given by

$$(2x + y - 3)^2 = c(y - x + 3).$$

21. It is convenient to use polar coordinates to formulate a model for this problem. Put the origin at the submarine at the time of sighting, and the polar axis the line from there to the destroyer at this time (the point $(9, 0)$). Initially the destroyer should steam at speed $2v$ directly toward the origin, until it reaches $(3, 0)$. During this time the submarine, moving at speed v, will have moved three units from the point where it was sighted. Let $\theta = \varphi$ be the ray (half-line) along which the submarine is moving.

    Upon reaching $(3, 0)$, the destroyer should execute a search pattern along a path $r = f(\theta)$. The object is to choose this path so that the sub and the destroyer both reach $(f(\varphi, \varphi)$ at the same time T after the sighting.

    From sighting to interception, the destroyer travels a distance

$$6 + \int_0^{\varphi} \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta,$$

    so

$$T = \frac{1}{2v}\left(6 + \int_0^{\varphi} \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta\right).$$

    For the submarine,

$$T = \frac{1}{v} f(\varphi).$$

    Equate these two expressions for T and differentiate with respect to φ to get

$$\frac{1}{2}\sqrt{(f(\varphi))^2 + (f'(\varphi))^2} = f'(\varphi).$$

    Denote the variable as θ and rearrange the last equation to obtain

$$\frac{f'(\theta)}{f(\theta)} = \pm\frac{1}{\sqrt{3}}.$$

    The positive sign here indicates that the destroyer should execute a starboard (left) turn, while the negative sign is for a portside turn. Taking the positive sign, solve for $f(\theta)$ to get

$$f(\theta) = ke^{\theta/\sqrt{3}}.$$

    Now $f(0) = k = 3$, so the path of the destroyer is part of the graph of

$$f(\theta) = 3se^{\theta/\sqrt{3}}.$$

    After sailing directly to $(3, 0)$, the destroyer should execute this spiral pattern. A similar conclusion follows if the negative sign of $1/\sqrt{3}$ is used.

    This shows that the destroyer can carry out a maneuver that will take it directly over the submarine at some time. However, there is no way to solve for the specific time, so it is unknown <u>when</u> this will occur.

Chapter 2

Second-Order Differential Equations

2.1 The Linear Second-Order Equation

1. It is a routine exercise in differentiation to show that $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous equation, while $y_p(x)$ is a solution of the nonhomogeneous equation. The Wronskian of $y_1(x)$ and $y_2(x)$ is

$$W(x) = \begin{vmatrix} \sin(6x) & \cos(6x) \\ 6\cos(6x) & -6\sin(6x) \end{vmatrix} = -6\sin^2(x) - 6\sin^2(x) = -6,$$

and this is nonzero for all x, so these solutions are linearly independent on the real line. The general solution of the nonhomogeneous differential equation is

$$y = c_1 \sin(6x) + c_2 \cos(6x) + \frac{1}{36}(x-1).$$

For the initial value problem, we need

$$y(0) = c_2 - \frac{1}{36} = -5$$

so $c_2 = -179/36$. And

$$y'(0) = 2 = 6c_1 + \frac{1}{36}$$

so $c_1 = 71/216$. The unique solution of the initial value problem is

$$y(x) = \frac{71}{216}\sin(6x) - \frac{179}{36}\cos(6x) + \frac{1}{36}(x-1).$$

3. The associated homogeneous equation has solutions e^{-2x} and e^{-x}. Their Wronskian is

$$W(x) = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix} = e^{-3x}$$

and this is nonzero for all x. The general solution of the nonhomogeneous differential equation is

$$y(x) = c_1 e^{-2x} + c_2 e^{-x} + \frac{15}{2}.$$

For the initial value problem, solve

$$y(0) = -3 = c_1 + c_2 + \frac{15}{2}$$

and

$$y'(0) = -1 = -2c_1 - c_2$$

to get $c_1 = 23/2, c_2 = -22$. The initial value problem has solution

$$y(x) = \frac{23}{2}e^{-2x} - 22e^{-x} + \frac{15}{2}.$$

5. The associated homogeneous equation has solutions

$$y_1(x) = e^x \cos(x), y_2(x) = e^x \sin(x).$$

These have Wronskian

$$W(x) = \begin{vmatrix} e^x \cos(x) & e^x \sin(x) \\ e^x \cos(x) - e^x \sin(x) & e^x \sin(x) + e^x \cos(x) \end{vmatrix} = e^{2x} \neq 0$$

so these solutions are independent. The general solution of the nonhomogeneous differential equation is

$$y(x) = c_1 e^x \cos(x) + c_2 e^x \sin(x) - \frac{5}{2}c^2 - 5x - \frac{5}{2}.$$

We need

$$y(0) = c_1 - \frac{5}{2} = 6$$

and

$$y'(0) = 1 = c_1 + c_2 - 5.$$

Solve these to get $c_1 = 17/2$ and $c_2 = -5/2$ to get the solution

$$y(x) = \frac{17}{2}e^x \cos(x) - \frac{5}{2}e^x \sin(x) - \frac{5}{2}x^2 - 5x - \frac{5}{2}.$$

7. The Wronskian of x^2 and x^3 is

$$W(x) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4.$$

Then $W(0) = 0$, while $W(x) \neq 0$ if $x \neq 0$. This is impossible if x^2 and x^3 are solutions of equation (2.2) for some functions $p(x)$ and $q(x)$. We conclude that these functions are not solutions of equation (2.2).

9. If $y_2(x)$ and $y_2(x)$ both have a relative extremum (max or min) at some x_0 within (a, b), then
$$y'(x_0) = y_2'(x_0) = 0.$$
But then the Wronskian of these functions vanishes at 0, and these solutions must be independent.

11. If $y_1(x_0) = y_2(x_0) = 0$, then the Wronskian of $y_1(x)$ and $y_2(x)$ is zero at x_0, and these two functions must be linearly dependent.

2.2 The Constant Coefficient Homogeneous Equation

1. From the differential equation we read the characteristic equation
$$\lambda^2 - \lambda - 6 = 0,$$
which has roots -2 and 3. The general solution is
$$y(x) = c_1 e^{-2x} + c_2 e^{3x}.$$

3. The characteristic equation is
$$\lambda^2 + 6\lambda + 9 = 0$$
with repeated roots $-3, -3$. Then
$$y(x) = c_1 e^{-3x} + c_2 x e^{-3x}$$
is a general solution.

5. characteristic equation $\lambda^2 + 10\lambda + 26 = 0$, with roots $-5 \pm i$; general solution
$$y(x) = c_1 e^{-5x} \cos(x) + c_2 e^{-5x} \sin(x).$$

7. characteristic equation $\lambda^2 + 3\lambda + 18 = 0$, with roots $-3/2 \pm 3\sqrt{7} i/2$; general solution
$$y(x) = c_2 e^{-3x/2} \cos\left(\frac{3\sqrt{7}x}{2}\right) + c_2 e^{-3x/2} \sin\left(\frac{3\sqrt{7}x}{2}\right).$$

9. characteristic equation $\lambda^2 - 14\lambda + 49 = 0$, with repeated roots $7, 7$; general solution
$$y(x) = e^{7x}(c_1 + c_2 x).$$

In each of Problems 11–20 the solution is found by finding a general solution of the differential equation and then using the initial conditions to find the particular solution of the initial value problem.

11. The differential equation has characteristic equation $\lambda^2 + 3\lambda = 0$, with roots $0, -3$. The general solution is

$$y(x) = c_1 + c_2 e^{-3x}.$$

Choose c_1 and c_2 to satisfy:

$$y(0) = c_1 + c_2 = 3,$$
$$y'(0) = -3c_2 = 6.$$

Then $c_2 = -2$ and $c_1 = 5$, so the unique solution of the initial value problem is

$$y(x) = 5 - 2e^{-3x}.$$

13. The initial value problem has the solution $y(x) = 0$ for all x. This can be seen by inspection or by finding the general solution of the differential equation and then solving for the constants to satisfy the initial conditions.

15. characteristic equation $\lambda^2 + \lambda - 12 = 0$, with roots $3, -4$. The general solution is

$$y(x) = c_1 e^{3x} + c_2 e^{-4x}.$$

We need

$$y(2) = c_1 e^6 + c_2 e^{-8} = 2$$

and

$$y'(2) = 3c_1 e^6 - 4c_2 e^{-8} = -1.$$

Solve these to obtain

$$c_1 = e^{-6}, c_2 = e^8.$$

The solution of the initial value problem is

$$y(x) = e^{-6} e^{3x} + e^8 e^{-4x}.$$

This can also be written

$$y(x) = e^{3(x-2)} + e^{-4(x-2)}.$$

17. $y(x) = e^{x-1}(29 - 17x)$

19.

$$y(x) = e^{(x+2)/2} \left[\cos(\sqrt{15}(x+2)/2) + \frac{5}{\sqrt{15}} \sin(\sqrt{15}(x+2)/2) \right]$$

21. (a) The characteristic equation is $\lambda^2 - 2\alpha\lambda + \alpha^2 = 0$, with α as a repeated root. The general solution is

$$y(x) = (c_1 + c_2 x)e^{\alpha x}.$$

(b) The characteristic equation is $\lambda^2 - 2\alpha\lambda + (\alpha^2 - \epsilon^2) = 0$, with roots $\alpha + \epsilon, \alpha - \epsilon$. The general solution is

$$y_\epsilon(x) = c_1 e^{(\alpha+\epsilon)x} + c_2 e^{(\alpha-\epsilon)x}.$$

We can also write

$$y_\epsilon(x) = \left(c_1 e^{\epsilon x} + c_2 e^{-\epsilon x}\right) e^{\alpha x}.$$

In general,

$$\lim_{\epsilon \to 0} y_\epsilon(x) = (c_1 + c_2)e^{\alpha x} \neq y(x).$$

Note, however, that the coefficients in the differential equations in (a) and (b) can be made arbitrarily close by choosing ϵ sufficiently small.

23. The roots of the characteristic equation are

$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} \text{ and } \lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}.$$

Because $a^2 - 4b < a^2$ by assumption, λ_1 and λ_2 are both negative (if $a^2 - 4b \geq 0$), or complex conjugates (if $a^2 - 4b < 0$). There are three cases.

Case 1 - Suppose λ_1 and λ_2 are real and unequal. Then the general solution is

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

and this has limit zero as $x \to \infty$ because λ_1 and λ_2 are negative.

Case 2 - Suppose $\lambda_1 = \lambda_2$. Now the general solution is

$$y(x) = (c_1 + c_2 x)e^{\lambda_1 x},$$

and this also has limit zero as $x \to \infty$.

Case 3 - Suppose λ_1 and λ_2 are complex. Now the general solution is

$$y(x) = \left[c_1 \cos(\sqrt{4b - a^2}\,x/2) + c_2 \sin(\sqrt{4b - a^2}\,x/2)\right] e^{-ax/2},$$

and this has limit zero as $x \to \infty$ because $a > 0$.

If, for example, $a = 1$ and $b = -1$, then one solution is $e^{(-1+\sqrt{5})x/2}$, and this tends to ∞ as $x \to \infty$.

2.3 Particular Solutions of the Nonhomogeneous Equation

1. Two independent solutions of $y'' + y = 0$ are $y_1(x) = \cos(x)$ and $y_2(x) = \sin(x)$, with Wronskian

$$W(x) = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = 1.$$

Let $f(x) = \tan(x)$ and use equations (2.7) and (2.8). First,

$$\begin{aligned} u_1(x) &= -\int \frac{y_2(x) f(x)}{W(x)} = -\int \tan(x)\sin(x)\, dx \\ &= -\int \frac{\sin^2(x)}{\cos(x)}\, dx \\ &= -\int \frac{1-\cos^2(x)}{\cos(x)}\, dx \\ &= \int \cos(x)\, dx - \int \sec(x)\, dx \\ &= \sin(x) - \ln|\sec(x) + \tan(x)|. \end{aligned}$$

Next,

$$\begin{aligned} u_2(x) &= \int \frac{y_1(x) f(x)}{W(x)}\, dx = \int \cos(x)\tan(x)\, dx \\ &= \int \sin(x)\, dx = -\cos(x). \end{aligned}$$

The general solution is

$$\begin{aligned} y(x) &= c_1\cos(x) + c_2\sin(x) + u_1(x)y_1(x) + u_2(x)y_2(x) \\ &= c_1\cos(x) + c_2\sin(x) - \cos(x)\ln|\sec(x) + \tan(x)|. \end{aligned}$$

For Problems 3–6, some details of the calculations are omitted.

3. The associated homogeneous equation has independent solutions $y_1(x) = \cos(3x)$ and $y_2(x) = \sin(3x)$, with Wronskian 3. The general solution is

$$y(x) = c_1\cos(3x) + c_2\sin(3x) + 4x\sin(3x) + \frac{4}{3}\cos(3x)\ln|\cos(3x)|.$$

5. $y_1(x) = e^x$ and $y_2(x) = e^{2x}$, with Wronskian $W(x) = e^{3x}$. With $f(x) = \cos(e^{-x})$, we find the general solution

$$y(x) = c_1 e^x + c_2 e^{2x} - e^{2x}\cos(e^{-x}).$$

In Problems 7–16 the method of undetermined coefficients is used to find a particular solution of the nonhomogeneous equation. Details are included for Problems 7 and 8, and solutions are outlined for the remainder of these problems.

7. The associated homogeneous equation has independent solutions $y_1(x) = e^{2x}$ and e^{-x}. Because $2x^2+5$ is a polynomial of degree 2, attempt a second degree polynomial
$$y_p(x) = Ax^2 + Bx + C$$
for the nonhomogeneous equation. Substitute $y_p(x)$ into this nonhomogeneous equation to obtain
$$2A - (2Ax + B) - 2(Ax^2 + Bx + C) = 2x^2 + 5.$$
Equating coefficients of like powers of x on the left and right, we have the equations
$$-2A = 2(\text{ coefficients of } x^2)$$
$$-2A - 2B - 0(\text{ coefficients of } x$$
$$2A - 2B - 2C = 5(\text{ constant term.})$$
Then $A = -1, B = 1$ and $C = -4$. Then
$$y_p(x) = -x^2 + x - 4$$
and a general solution of the (nonhomogeneous) equation is
$$y = c_1e^{2x} + c_2e^{-x} - x^2 + x - 4.$$

9. $y_1(x) = e^x \cos(3x)$ and $y_2(x) = e^x \sin(3x)$ are independent solutions of the associated homogeneous equation. Try a particular solution $y_p(x) = Ax^2 + Bx + C$ to obtain the general solution
$$y(x) = c_1e^x \cos(3x) + c_2e^x \sin(3x) + 2x^2 + x - 1.$$

11. For the associated homogeneous equation, $y_1(x) = e^{2x}$ and $y_2(x) = e^{4x}$. Because e^x is not a solution of the homogeneous equation, attempt a particular solution of the nonhomogeneous equation of the form $y_p(x) = Ae^x$. We get $A = 1$, so a general solution is
$$y(x) = c_1e^{2x} + c_2e^{4x} + e^x.$$

13. $y_1(x) = e^x$ and $y_2(x) = e^{2x}$. Because $f(x) = 10\sin(x)$, attempt
$$y_p(x) = A\cos(x) + B\sin(x).$$
Substitute this into the (nonhomogeneous) equation to find that $A = 3$ and $B = 1$. A general solution is
$$y(x) = c_1e^x + c_2e^{2x} + 3\cos(x) + \sin(x).$$

15. $y_1(x) = e^{2x}\cos(3x)$ and $y_2(x) = e^{2x}\sin(3x)$. Try

$$y_p x = Ae^{2x} + Be^{3x}.$$

This will work because neither e^{2x} nor e^{3x} is a solution of the associated homogeneous equation. Substitute $y_p(x)$ into the differential equation and obtain $A = 1/3, B = -1/2$. The differential equation has general solution

$$y(x) = [c_1\cos(3x) + c_2\sin(3x)]e^{2x} + \frac{1}{3}e^{2x} - \frac{1}{2}e^{3x}.$$

In Problems 17–24 the strategy is to first find a general solution of the differential equation, then solve for the constants to find a solution satisfying the initial conditions. Problems 17–22 are well suited to the use of undetermined coefficients, while Problems 23 and 24 can be solved fairly directly using variation of parameters.

17. $y_1(x) = e^{2x}$ and $y_2(x) = e^{-2x}$. Because e^{2x} is a solution of the associated homogeneous equation, use xe^{2x} in the method of undetermined coefficients, attempting

$$y_p(x) = Axe^{2x} + Bx + C.$$

Substitute this into the nonhomogeneous differential equation to obtain

$$4Axe^{2x} + 4Axe^{2x} - 4Axe^{2x} - 4Bx - 4C = -7e^{2x} + x.$$

Then $A = -7/4, B = -1/4$ and $C = 0$, so the differential equation has the general solution

$$y(x) = c_1e^{2x} + c_2e^{-2x} - \frac{7}{4}xe^{2x} - \frac{1}{4}x.$$

We need

$$y(0) = c_1 + c_2 = 1$$

and

$$y'(0) = 2c_1 - 2c_2 - \frac{7}{4} - \frac{1}{4} = 3.$$

Then $c_1 = 7/4$ and $c_2 = -3/4$. The initial value problem has the unique solution

$$y(x) = \frac{7}{4}e^{2x} - \frac{3}{4}e^{-2x} - \frac{7}{4}xe^{2x} - \frac{1}{4}x.$$

19. We find the general solution

$$y(x) = c_1e^{-2x} + c_2e^{-6x} + \frac{1}{5}e^{-x} + \frac{7}{12}.$$

The solution of the initial value problem is

$$y(x) = \frac{3}{8}e^{-2x} - \frac{19}{120}e^{-6x} + \frac{1}{5}e^{-x} + \frac{7}{12}.$$

21. e^{4x} and e^{-2x} are independent solutions of the associated homogeneous equation. The nonhomogeneous equation has general solution

$$y(x) = c_1e^{4x} + c_2e^{-2x} - 2e^{-x} - e^{2x}.$$

The solution of the initial value problem is

$$y(x) = 2e^{4x} + 2e^{-2x} - 2e^{-x} - e^{2x}.$$

23. The differential equation has general solution

$$y(x) = c_1e^{x} + c_2e^{-x} - \sin^2(x) - 2.$$

The solution of the initial value problem is

$$y(x) = 4e^{-x} - \sin^2(x) - 2.$$

2.4 The Euler Differential Equation

Details are included with solutions for Problems 1–2, while just the solutions are given for Problems 3–10. These solutions are for $x > 0$.

1. Read from the differential equation that the characteristic equation is

$$r^2 + r - 6 = 0$$

with roots $2, -3$. The general solution is

$$y(x) = c_1x^2 + c_2x^{-3}.$$

3.

$$y(x) = c_1\cos(2\ln(x)) + c_2\sin(2\ln(x))$$

5.

$$y(x) = c_1x^2 + c_1\frac{1}{x^4}$$

7.

$$y(x) = c_1\frac{1}{x^2} + c_2\frac{1}{x^3}$$

9.

$$y(x) = \frac{1}{x^{12}}(c_1 + c_2\ln(x))$$

11. The general solution of the differential equation is

$$y(x) = c_1x^3 + c_2x^{-7}.$$

From the initial conditions, we need

$$y(2) = 8c_1 + 2^{-7}c_2 = 1 \text{ and } y'(2) = 3c_12^2 - 7c_22^{-8} = 0.$$

Solve for c_1 and c_2 to obtain the solution of the initial value problem

$$y(x) = \frac{7}{10}\left(\frac{x}{2}\right)^3 + \frac{3}{10}\left(\frac{x}{2}\right)^{-7}.$$

13. $y(x) = x^2(4 - 3\ln(x))$

15. $y(x) = 3x^6 - 2x^4$

17. With $Y(t) = y(e^t)$, use the chain rule to get

$$y'(x) = \frac{dY}{dt}\frac{dt}{dx} = \frac{1}{x}Y'(t)$$

and then

$$\begin{aligned} y''(x) &= \frac{d}{dx}\left(\frac{1}{x}Y'(t)\right) \\ &= -\frac{1}{x^2}Y'(t) + \frac{1}{x}\frac{d}{dx}(Y'(t)) \\ &= -\frac{1}{x^2}Y'(t) + \frac{1}{x}\frac{dY'}{dt}\frac{dt}{dx} \\ &= -\frac{1}{x^2}Y'(t) + \frac{1}{x}\frac{1}{x}Y''(t) \\ &= \frac{1}{x^2}(Y''(t) - Y'(t)). \end{aligned}$$

Then

$$x^2y''(x) = Y''(t) - Y'(t).$$

Substitute these into Euler's equation to get

$$Y''(t) + (A - 1)Y'(t) + BY(t) = 0.$$

This is a constant coefficient second-order homogeneous differential equation for $Y(t)$, which we know how to solve.

19. The problem to solve is

$$x^2y'' - 5dxy' + 10y = 0; y(1) = 4, y'(1) = -6.$$

We know how to solve this problem. Here is an alternative method, using the transformation $x = e^t$, or $t = \ln(x)$ for $x > 0$ (since the initial conditions are specified at $x = 1$). Euler's equation transforms to

$$Y'' - 6Y' + 10Y = 0.$$

However, also transform the initial conditions:

$$Y(0) = y(1) = 4, Y'(0) = (1)y'(1) = -6.$$

This differential equation for $Y(t)$ has general solution

$$Y(t) = c_1e^{3t}\cos(t) + c_2e^{3t}\sin(t).$$

Now

$$Y(0) = c_2 = 4$$

and

$$Y'(0) = 3c_1 + c_2 = -6,$$

so $c_2 = -18$. The solution of the transformed initial value problem is

$$Y(t) = 4e^{3t}\cos(t) - 18e^{3t}\sin(t).$$

The original initial value problem therefore has the solution

$$y(x) = 4x^3\cos(\ln(x)) - 19x^3\sin(\ln(x))$$

for $x > 0$. The new twist here is that the entire initial value problem (including initial conditions) was transformed in terms of t and solved for $Y(t)$, then this solution $Y(t)$ in terms of t was transformed back to the solution $y(x)$ in terms of x.

2.5 Series Solutions

2.5.1 Power Series Solutions

1. Put $y(x) = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation to obtain

$$\begin{aligned} y' - xy &= \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=2}^{\infty} a_{n-2} x^{n-1} \\ &= a_1 + (2a_2 - a_0)x + \sum_{n=3}^{\infty}(n a_n - a_{n-2})x^{n-1} \\ &= 1 - x. \end{aligned}$$

Then a_0 is arbitrary, $a_1 = 1$, $2a_2 - a_0 = -1$, and

$$a_n = \frac{1}{n} a_{n-2} \text{ for } n = 3, 4, \cdots.$$

This is the recurrence relation. If we set $a_0 = c_0 + 1$, we obtain the coefficients

$$a_2 = \frac{1}{2}c_0, a_4 = \frac{1}{2 \cdot 4}c_0, a_6 = \frac{1}{2 \cdot 4 \cdot 6}c_0,$$

and so on. Further,

$$a_1 = 1, a_3 = \frac{1}{3}, a_5 = \frac{1}{3 \cdot 5}, a_7 = \frac{1}{3 \cdot 5 \cdot 7}$$

and so on. The solution can be written

$$\begin{aligned} y(x) = 1 &+ \sum_{n=0}^{\infty} \frac{1}{3 \cdot 5 \cdots 2n+1} x^{2n+1} \\ &+ c_0 \left(1 + \sum_{n=1}^{\infty} \frac{1}{2 \cdot 4 \cdots 2n} x^{2n}\right). \end{aligned}$$

3. Write

$$\begin{aligned} y' + (1-x^2)y &= \sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+2} \\ &= (a_1 + a_0) + (2a_2 + a_1)x + \sum_{n=3}^{\infty}(na_n + a_{n-1} - a_{n-3})x^{n-1} \\ &= x. \end{aligned}$$

The recurrence relation is

$$na_n + a_{n-1} - a_{n-3} = 0 \text{ for } n = 3, 4, \cdots .$$

Here a_0 is arbitrary, $a_1 + a_0 = 0$ and $2a_2 + a_1 = 1$. This gives us the solution

$$\begin{aligned} y(x) = a_0 &\left(1 - x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 - \frac{7}{4!}x^4 + \frac{19}{5!}x^5 + \cdots\right) \\ &+ \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{11}{5!}x^5 - \frac{31}{6!}x^6 + \cdots . \end{aligned}$$

5. Write

$$\begin{aligned} y'' - xy' + y &= \sum_{n=2}^{\infty} n(n-1)_n x^{n-2} - \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n \\ &= 2a_2 + a_0 + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n = 3. \end{aligned}$$

Here a_0 and a_1 are arbitrary and $a_2 = (3 - a_0)/2$. The recurrence relation is

$$a_{n+2} = \frac{n-1}{(n+2)(n+1)} \text{ for } n = 1, 2, \cdots .$$

This yields the general solution

$$\begin{aligned} y(x) &= a_0 + a_1 x + \frac{3-a_0}{2}x^2 + \frac{3-a_0}{4!}x^4 \\ &+ \frac{3(3-a_0)}{6!}x^6 + \frac{3 \cdot 5(3-a_0)}{8!}x^8 + \cdots . \end{aligned}$$

7. We have

$$\begin{aligned} y'' - x^2 y' + 2y &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \\ &- \sum_{n=1}^{\infty} na_n x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^n \\ &= 2a_2 + 2a_0 + (6a_3 + 2a_1)x \\ &+ \sum_{n=1}^{\infty}(n(n-1)a_n - (n-3)a_{n-3} + 2a_{n-2})x^{n-2} = x. \end{aligned}$$

Then a_0 and a_1 are arbitrary, $a_2 = -a_0$, and $6a_3+2a_1 = 1$. The recurrence relation is

$$a_n = \frac{(n-3)a_{n-3} - 2a_{n-2}}{n(n-1)}$$

for $n = 4, 5, \cdots$. The general solution has the form

$$\begin{aligned} y(x) &= a_0\left[1 - x^2 + \frac{1}{6}x^4 - \frac{1}{10}x^5 - \frac{1}{90}x^6 + \cdots\right] \\ &+ a_1\left[x - \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{30}z^5 - \frac{7}{180}x^6 + \cdots\right] \\ &+ \frac{1}{6}x^3 - \frac{1}{6}x^5 + \frac{1}{60}x^6 + \frac{1}{1260}x^7 - \frac{1}{480}x^8 + \cdots . \end{aligned}$$

Note that $a_0 = y(0)$ and $a_1 = y'(0)$. The third series represents the solution obtained subject to $y(0) = y'(0) = 0$.

9. We have

$$\begin{aligned} &y'' + (1-x)y' + 2y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \\ &+ \sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=1}^{\infty} na_n x^n + 2\sum_{n=0}^{\infty} a_n x^n \\ &= (2a_2 + a_1 + 2a_0) + \sum_{n=3}^{\infty}(n(n-1)a_n + (n-1)a_{n-1} - (n-4)a_{2n-2})x^{n-2} \\ &= 1 - x^2. \end{aligned}$$

Then a_0 and a_1 are arbitrary, $2a_2 + a_1 + 2a_0 = 1$, $6a_3 + 2a_2 + a_1 = 0$, and $12a_4 + 3a_3 = -1$. The recurrence relation is

$$a_n = \frac{-(n-1)a_{n-1} + a_{n-4}a_{n-2}}{n(n-1)}$$

for $n = 5, 6, \cdots$. The general solution is

$$\begin{aligned} y(x) &= a_0\left[1 - x^2 + \frac{1}{3}x^3 - \frac{1}{12}x^4 + \frac{1}{30}x^5 - \cdots\right] \\ &+ a_1\left(x - \frac{1}{2}x^2\right) + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{360}x^6 + \frac{1}{2520}x^7 + \cdots . \end{aligned}$$

Here $a_0 = y(0)$ and $a_1 = y'(0)$.

2.5.2 Frobenius Solutions

1. Substitute $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation to get

$$\begin{aligned} xy'' + (1-x)y' + y &= \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} \\ &+ \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} \\ &= r^2 c_0 x^{r-1} + \sum_{n=1}^{\infty}((n+r)^2 c_n - (n+r-2)c_{n-1})x^{n+r-1} \\ &= 0. \end{aligned}$$

Because c_0 is assumed to be nonzero, r must satisfy the indicial equation $r^2 = 0$, so $r_1 = r_2 = 0$. One solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^n,$$

while a second solution has the form

$$y_2(x) = y_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^n.$$

For the first solution, choose the coefficients to satisfy $c_0 = 1$ and

$$c_n = \frac{n-2}{n^2} c_{n-1} \text{ for } n = 1, 2, \cdots .$$

This yields the solution $y_1(x) = 1 - x$. The second solution is therefore

$$y_2(x) = (1-x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^n.$$

Substitute this into the differential equation to obtain

$$\begin{aligned} & x\left[-\frac{2}{x} - \frac{1-x}{x^2}\right] + (1-x)\left[-\ln(x) + \frac{1-x}{x}\right] \\ &+ (1-x)\ln(x) + \sum_{n=2}^{\infty} n(n-1)c_n^* x^{n-1} + (1-x)\sum_{n=1}^{\infty} c_n^* x^{n-1} \\ &+ \sum_{n=1}^{\infty} c_n^* x^n \\ &= (-3 + c_1^*) + (1 + 4c_2^*)x + \sum_{n=3}^{\infty}(n^2 c_n^* - (n-2)c_{n-1}^*)x^{n-2} \\ &= 0. \end{aligned}$$

The coefficients are determined by $c_1^* = 3$, $c_2^* = -1/4$, and

$$c_n^* = \frac{n-2}{n^2} \text{ for } n = 3, 4, \cdots .$$

A second solution is

$$y_2(x) = (1-x)\ln(x) + 3x - \sum_{n=2}^{\infty} \frac{1}{n(n-1)} x^n.$$

3. The indicial equation is $r^2 - 4r = 0$, so $r_1 = 4$ and $r_2 = 0$. There are solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+4} \text{ and } y_2(x) = k y_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^n.$$

With $r = 4$ the recurrence relation is

$$c_n = \frac{n+1}{n} c_{n-1} \text{ for } n = 1, 2, \cdots .$$

Then

$$y_1(x) = x^4(1 + 2x + 3x^2 + 4x^3 + \cdots).$$

Using the geometric series, we can observe that

$$\begin{aligned} y_1(x) &= x^4 \frac{d}{dx}(1 + x + x^2 + x^3 + \cdots) \\ &= x^4 \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{x^4}{(1-x)^2}. \end{aligned}$$

This gives us the second solution

$$y_2(x) = \frac{3-4x}{(1-x)^2}.$$

5. The indicial equation is $4r^2 - 2r = 0$, with roots $r_1 = 1/2$ and $r_2 = 0$. There are solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+1/2} \text{ and } y_2(x) = \sum_{n=0}^{\infty} c_n^* x^n.$$

Substitute these into the differential equation to get

$$\begin{aligned} y_1(x) &= x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!(3 \cdot 5 \cdot 7 \cdots (2n+1))} x^n\right] \\ &= x^{1/2}\left[1 - \frac{1}{6}x + \frac{1}{120}x^2 - \frac{1}{5040}x^3 + \frac{1}{362880}x^4 + \cdots\right] \end{aligned}$$

and

$$\begin{aligned} y_2(x) &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!(1 \cdot 3 \cdot 5 \cdots (2n-1))} x^n \\ &= 1 - \frac{1}{2}x + \frac{1}{24}x^2 - \frac{1}{720}x^3 + \frac{1}{40320}x^4 + \cdots . \end{aligned}$$

7. The indicial equation is $r^2 - 3r + 2 = 0$, with roots $r_1 = 2$ and $r_2 = 1$. There are solutions

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+2} \text{ and } \sum_{n=0}^{\infty} c_n^* x^{n-2}.$$

Substitute these in turn into the differential equation to obtain the solutions

$$y_1(x) = x^2 + \frac{1}{3!}x^4 + \frac{1}{5!}x^6 + \frac{1}{7!}x^8 + \cdots$$

and

$$y_2(x) = x - x^2 + \frac{1}{2!}x^3 - \frac{1}{3!}x^4 + \frac{1}{4!}x^5 - \cdots .$$

We can recognize these series as

$$y_1(x) = x \sinh(x) \text{ and } y_2(x) = xe^{-x}.$$

9. The indicial equation is $2r^2 = 0$, with roots $r_1 = r_2 = 0$. There are solutions

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^n \text{ and } y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} c_n^* x^n.$$

Upon substituting these into the differential equation, we obtain the independent solutions

$$y_1(x) = 1 - x$$

and

$$y_2(x) = (1 - x)\ln\left(\frac{x}{x-2}\right) - 2.$$

Chapter 3

The Laplace Transform

3.1 Definition and Notation

1.
$$F(s) = \frac{3(s^2-4)}{(s^2+4)^2}$$

3.
$$H(s) = \frac{14}{s^2} - \frac{7}{s^2+49}$$

5.
$$K(s) = -\frac{10}{(s+4)^2} + \frac{3}{s^2+9}$$

7. $q(t) = \cos(8t)$

9.
$$p(t) = e^{-42t} - \frac{1}{6}t^3e^{-3t}$$

11. (a) From the definition,
$$F(s) = \mathcal{L}[f](s) = \lim_{R\to\infty}\int_0^R e^{-st}f(t)\,dt.$$
For each R, let N be the largest integer such that $(N+1)T \le R$ to write
$$\int_0^R e^{-st}f(t)\,dt = \sum_{n=0}^{N}\int_{nT}^{(n+1)T} e^{-st}f(t)\,dt + \int_{(N+1)T}^{R} e^{-st}f(t)\,dt.$$
By choosing R sufficiently large, we can make the last integral on the right as small as we like. Further, $R\to\infty$ as $N\to\infty$, so
$$\int_0^\infty e^{-st}f(t)\,dt = \sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} e^{-st}f(t)\,dt.$$

(b) Use the periodicity of $f(t)$ and the change of variables $u = t - nT$ to write

$$\int_{nT}^{(n+1)T} e^{-st} f(t)\,dt = \int_0^T e^{-s(u+nT)} f(u+nT)\,dt$$
$$= e^{-snT} \int_0^T e^{-su} f(u)\,du,$$

because $f(u + nT) = f(u)$.

(c) Use the results of (a) and (b) to write

$$\mathcal{L}[f](s) = \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st} f(t)\,dt$$
$$= \sum_{n=0}^{\infty} e^{-snT} \int_0^T f(t)\,dt$$
$$= \left[\sum_{n=0}^{\infty} e^{-snT}\right] \int_0^T e^{-st} f(t)\,dt,$$

because the summation is independent of t.

(d) For $s > 0$, $0 < e^{-st} < 1$ and we can use the geometric series to obtain

$$\sum_{n=0}^{\infty} e^{-snt} = \sum_{n=0}^{\infty} \left(e^{-sT}\right)^n = \frac{1}{1 - e^{-sT}}.$$

Therefore

$$\mathcal{L}[f](s) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\,dt.$$

13. f has period $T = \pi/\omega$ and

$$\int_0^T e^{-st} f(t)\,dt = \int_0^{\pi/\omega} E e^{-st} \sin(\omega t)\,dt$$
$$= \frac{E\omega}{s^2 + \omega^2} \frac{(1 + e^{-\pi s/\omega})}{1 - e^{-\pi s/\omega}}.$$

15. f has period 6 and $f(t) = t/3$ for $0 \le t < 6$. Compute

$$\int_0^T e^{-st} f(t)\,dt = \int_0^6 \frac{1}{3} t e^{-st}\,dt = \frac{1}{3s^2}(1 - 6se^{-6s} - e^{-6s}).$$

Then

$$\mathcal{L}[f](s) = \frac{1}{3s^2} \frac{1 - 6se^{-6s} - e^{-6s}}{1 - e^{-6s}}.$$

17. Here

$$f(t) = \begin{cases} h & \text{for } 0 < t \le a, \\ 0 & \text{for } a < t \le 2a, \end{cases}$$

with period $2a$. Then

$$\int_0^{2a} e^{-st} f(t)\,dt = \int_0^a he^{-st}\,dt = \frac{h}{s}(1 - e^{-as}).$$

Then

$$\mathcal{L}[f](s) = \frac{h}{s}\frac{1 - e^{-as}}{1 - e^{-2as}}.$$

3.2 Solution of Initial Value Problems

In many of these problems a partial fractions decomposition is used to find the inverse transform of the transform of the solution (hence find the solution). Partial fractions are reviewed in a web module.

1. Transform the differential equation, using the operational formula, to obtain

$$sY(s) - y(0) + 4Y(s) = \frac{1}{s}.$$

With $y(0) = 3$, this is

$$sY - 3 + 4Y = \frac{1}{s}.$$

Then

$$Y(s) = \frac{1}{s+4}\left[\frac{1}{s} - 3\right] = \frac{1 - 3s}{s(s+4)}.$$

Decompose this into a sum of simpler fractions:

$$Y(s) = \frac{A}{s} + \frac{B}{s+4}.$$

It these fractions are added, the numerator must equal the numerator of the original fraction, $1 - 3s$:

$$A(s+4) + Bs = 1 - 3s.$$

Then

$$(A+B)s + 4A = 1 - 3s.$$

Matching coefficients of like powers of x, this requires that

$$A + B = -3 \text{ and } 4A = 1.$$

Then $A = 1/4$ and $B = -13/4$. Now

$$Y(s) = \frac{1}{4}\frac{1}{s} - \frac{13}{4}\frac{1}{s+4}.$$

Now we immediately read from Table 3.1 that

$$y(t) = \mathcal{L}^{-1}[Y](t) = \frac{1}{4} - \frac{13}{4}e^{-4t}.$$

This is the solution of the initial value problem.

3. Take the transform of the differential equation and insert the initial data and solve for $Y(s)$ to get

$$Y(s) = \frac{1}{s+4}\left(\frac{s}{s^2+1}\right).$$

Use a partial fractions decomposition to obtain

$$Y(s) = -\frac{4}{17}\frac{1}{s+4} + \frac{1}{17}\frac{4s+1}{s^2+1}.$$

The inverse of this is the solution:

$$y(t) = -\frac{4}{17}e^{-4t} + \frac{4}{17}\cos(t) + \frac{1}{17}\sin(t).$$

5. The transform of the initial value problem is

$$sY - 4 - 2Y = \frac{1}{s} - \frac{1}{s^2}.$$

Then

$$\begin{aligned} Y(s) &= \frac{1}{s-2}\left(\frac{1}{s} - \frac{1}{s^2} + 4\right) \\ &= \frac{1}{2}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s} + \frac{17}{4}\frac{1}{s-2}. \end{aligned}$$

The solution is

$$y(t) = \frac{1}{2}t - \frac{1}{4} + \frac{17}{4}e^{2t}.$$

7. Transform the differential equation to obtain

$$s^2Y - sy(0) - y'(0) - 4(sY - y(0)) + 4Y = \frac{s}{s^2+1}.$$

Insert the initial conditions to get

$$(s-2)^2 Y = \frac{s}{s^2+1} + s - 5.$$

Then

$$Y(s) = \frac{s}{(s^2+1)(s-2)^2} + \frac{s-5}{(s-2)^2}.$$

With a some manipulation, write this as

$$Y(s) = \frac{s^3 - 5s^2 + 2s - 5}{(s^2+1)(s-2)^2}.$$

Expand this in partial fractions:

$$Y(s) = \frac{As+B}{s^2+1} + \frac{C}{s-2} + \frac{D}{(s-2)^2}.$$

To determine the coefficients, we need the numerator in the sum of these fractions to equal the numerator in $Y(s)$:

$$\begin{aligned}
&(As+B)(s-2)^2 + C(s-2)(s^2+1) + D(s^2+1) \\
&= (A+C)s^3 + (-4A+B-2C+D)s^2 + (4A-4B+C)s + (4B-2C+D) \\
&= s^3 - 5s^2 + 2s - 5.
\end{aligned}$$

Matching coefficients of powers of s, we obtain:

$$\begin{aligned}
A + C &= 1, \\
-4A + B - 2C + D &= -5, \\
4A - 4B + C &= 2, \\
4B - 2C + D &= -5.
\end{aligned}$$

Then

$$A = \frac{3}{25}, B = -\frac{4}{25}, C = \frac{22}{25}, D = -\frac{13}{5}.$$

Then

$$Y(s) = \frac{3}{25}\frac{s}{s^2+1} - \frac{4}{25}\frac{1}{s^2+1} + \frac{22}{25}\frac{1}{s-2} - \frac{13}{5}\frac{1}{(s-2)^2}.$$

These terms are easily inverted to obtain the solution

$$y(t) = \frac{3}{25}\cos(t) - \frac{4}{25}\sin(t) + \frac{22}{25}e^{2t} - \frac{13}{5}te^{2t}.$$

9. Upon transforming the differential equation and inserting the initial conditions, we have

$$\begin{aligned}
Y(s) &= \frac{1}{s^2+16}\left[\frac{1}{s} + \frac{1}{s^2} - 2s + 1\right] \\
&= \frac{1}{16}\frac{1}{s^2} + \frac{1}{16}\frac{1}{s} - \frac{33}{16}\frac{s}{s^2+16} - \frac{15}{64}\frac{1}{s^2+16}.
\end{aligned}$$

The solution is

$$y(t) = \frac{1}{16}(1+t) - \frac{33}{16}\cos(4t) + \frac{15}{64}\sin(4t).$$

11. Begin with the definition of the Laplace transform and integrate by parts:

$$\begin{aligned}
\mathcal{L}[f'](s) &= \int_0^\infty e^{-st} f'(t)\,dt \\
&= e^{-st} f(t)\Big]_a^b - \int_0^\infty -se^{-st} f(t)\,dt \\
&= -f(0) + s\int_0^\infty e^{-st} f(t)\,dt \\
&= sF(s) - f(0).
\end{aligned}$$

3.3 The Heaviside Function and Shifting Theorems

In each of Problems 1–15, we will indicate shifting $f(t)$ by a, replacing t with $t-a$, by writing

$$[f(t)]_{t\to t-a}.$$

Similarly, if we replace s with $s-a$ in the transform $F(s)$ of $f(t)$, we will write

$$[F(s)]_{s\to s-a}$$

or sometimes

$$\mathcal{L}[f(t)]_{s\to s-a}.$$

This notation is sometimes useful in applying a shifting theorem or inverse shifting theorem.

1. Apply the shifting theorem:

$$\begin{aligned}
\mathcal{L}[(t^3-3t+2)e^{-2t}](s) &= \mathcal{L}[t^3-3t+2]_{s\to s+2} \\
&= \frac{6}{(s+2)^4} - \frac{3}{(s+2)^2} + \frac{2}{s+2}.
\end{aligned}$$

3. First write

$$\begin{aligned}
f(t) &= [1-H(t-7)] + H(t-7)\cos(t) \\
&= [1-H(t-7)] + H(t-7)\cos((t-7)+7) \\
&= [1-H(t-7)] + \cos(7)H(t-7)\cos(t-7) - \sin(7)H(t-7)\sin(t-7).
\end{aligned}$$

Then

$$\mathcal{L}[f](s) = \frac{1}{s}\left(1-e^{-7s}\right) + \frac{s}{s^2+1}\cos(7)e^{-7s} - \frac{1}{s^2+1}\sin(7)e^{-7s}.$$

5. First, write the function as

$$\begin{aligned} f(t) &= t + (1-4t)H(t-3) \\ &= t + (1-4(t-3)+3)H(t-3) \\ &= t - 11H(t-3) - 4(t-3)H(t-3). \end{aligned}$$

Then

$$\mathcal{L}[f](s) = \frac{1}{s^2} - \frac{11}{s}e^{-3s} - \frac{4}{s^2}e^{-3s}.$$

7. Replace s with $s+1$ in the transform of $1 - t^2 + \sin(t)$ to get

$$\mathcal{L}[f](s) = \frac{1}{s+1} - \frac{2}{(s+1)^2} + \frac{1}{(s+1)^2+1}.$$

9. First, write

$$f(t) = (1 - H(t-2\pi))\cos(t) + H(t-2\pi)(2-\sin(t)).$$

Then

$$\updownarrow[f](s) = \frac{s}{s^2+1} + \left(\frac{2}{s} - \frac{s}{s^2+1} - \frac{1}{s^2+1}\right)e^{-2\pi s}.$$

11. Because

$$\mathcal{L}[t\cos(t)](s) = \frac{s^2-9}{(s^2+9)^2},$$

we obtain the transform of $te^{-t}\cos(t)$ by replacing s with $s+1$:

$$\mathcal{L}[te^{-t}\cos(t)](s) = \frac{(s+1)^2-9}{((s+1)^2+9)^2}.$$

13. First, put $f(t)$ in terms of the Heaviside function as

$$\begin{aligned} f(t) &= (1 - H(t-16))(t-2) - H(t-16) \\ &= t - 2 + (1-t)H(t-16). \end{aligned}$$

Then

$$\mathcal{L}[f](s) = \frac{1}{s^2} - \frac{2}{s} + \left(\frac{1}{s} - \frac{1}{s^2}\right)e^{-16s}.$$

15. Replace s with $s+5$ in the transform of $t^4 + 2t^2 + 1$ to obtain

$$F(s) = \frac{24}{(s+5)^5} + \frac{4}{(s+5)^3} + \frac{1}{(s+5)^2}.$$

17. Write

$$F(s) = \frac{1}{(s-2)^2+1}.$$

This is the transform of $\sin(t)$ with s replaced by $s-2$. Therefore

$$f(t) = e^{2t}\sin(t).$$

19. Because $3/(s^2+9)$ is the transform of $\sin(3t)$, then
$$f(t) = \frac{1}{3}\sin(3(t-2))H(t-2).$$

21. We recognize that
$$F(s) = \frac{1}{(s+3)^2 - 2}$$
so
$$f(t) = \frac{1}{\sqrt{2}}e^{-3t}\sinh(\sqrt{2}t).$$

23. Write
$$F(s) = \frac{(s+3)-1}{(s+3)^2-8}$$
to obtain
$$f(t) = e^{-3t}\cosh(2\sqrt{2}t) - \frac{1}{2\sqrt{2}}e^{-3t}\sinh(2\sqrt{2}t).$$

25. First use a partial fractions decomposition to write
$$\frac{1}{s(s^2+16)} = \frac{1}{16}\frac{1}{s} - \frac{1}{16}\frac{s}{s^2+16}.$$
From this,
$$f(t) = \frac{1}{16}(1-\cos(4(t-21)))H(t-21).$$

27. The initial value problem is
$$y'' + 4y = 3H(t-4); y(0) = 1, y'(0) = 0.$$
This transforms to
$$Y(s) = \frac{3}{4}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]e^{-4s} + \frac{s}{s^2+4}.$$
Invert this to obtain the solution
$$y(t) = \cos(2t) + \frac{3}{4}(1-\cos(2(t-4)))H(t-4).$$

29. The problem is
$$y^{(3)} - 8y' = 2H(t-6); y(0) = y'(0) = 0.$$
The transform of the problem is
$$Y(s) = \left[-\frac{1}{4s} + \frac{1}{12}\frac{1}{s-2} + \frac{1}{6}\frac{s}{s^2+2s-4}\right]e^{-6s}.$$
Invert this to obtain the solution
$$y(t) = \left[-\frac{1}{4} + \frac{1}{12}e^{-2(t-6)} + \frac{1}{6}e^{-(t-6)}\cos(\sqrt{3}(t-6))\right]H(t-6).$$

31. The problem is

$$y^{(3)} - y'' + 4y' - 4y = 1 + H(t-5); y(0) = y'(0) = 0, y'' = 1.$$

Transform this to obtain

$$Y(s) = \left[-\frac{1}{4s} + \frac{2}{5}\frac{1}{s-1} - \frac{3}{20}\frac{s}{s^2+4} - \frac{2}{5}\frac{1}{s^2+4} \right] (1 - e^{-5s}).$$

Invert this for the solution

$$\begin{aligned} y(t) = &-\frac{1}{4} + \frac{2}{5}e^t - \frac{3}{20}\cos(2t) - \frac{1}{5}\sin(2t) \\ &- \left[-\frac{1}{4} + \frac{2}{5}e^{t-5} - \frac{3}{20}\cos(2(t-5)) - \frac{1}{5}\sin(2(t-5)) \right] H(t-5). \end{aligned}$$

33. The current $i(t)$ is modeled by

$$Li' + Ri = k(1 - H(t-5)); i(0) = 0.$$

Transform the problem and solve for $I(s)$ to get

$$\begin{aligned} I(s) &= \frac{k}{Ls+R}(1 - e^{-5s}) \\ &= \frac{k}{R}\left[\frac{1}{s} - \frac{1}{s+R/L} \right] (1 - e^{-5s}). \end{aligned}$$

Invert this to obtain the solution for the current:

$$i(t) = \frac{k}{R}(1 - e^{-Rt/L}) - \frac{k}{R}(1 - e^{-R(t-5)/L})H(t-5).$$

In Problems 34–38, the Heaviside formula is used to compute an inverse transform. One way to use this formula efficiently is to begin with $F(s)$, which has the form

$$F(s) = \frac{p(s)}{(s-a_1)(s-a_2)\cdots(s-a_n)}.$$

For the first term, cover up the $s - a_1$ factor and evaluate the resulting quotient at a_1 to get the coefficient of $e^{a_1 t}$. Then cover up the $s - a_2$ factor and evaluate the resulting quotient at a_2 for the coefficient of $e^{a_2 t}$. Continue this through the n simple zeros of the denominator to obtain $f(t)$.

35. With

$$F(s) = \frac{s^2}{(s-1)(s-2)(s+5)}$$

we have $a_1 = 1, a_2 = 2$ and $a_3 = -5$. Then

$$\begin{aligned} f(t) &= \frac{1^2}{(-1)(6)}e^t + \frac{2^2}{(1)(7)}e^{2t} + \frac{5^2}{(-6)(-7)}e^{5t} \\ &= \frac{1}{6}e^t + \frac{4}{7}e^{2t} + \frac{25}{42}e^{5t}. \end{aligned}$$

37. Here

$$F(s) = \frac{s^2+2s-1}{(s-3)(s-5)(s+8)}.$$

Then

$$\begin{aligned} f(t) &= \frac{14}{(-2)(11)}e^{3t} + \frac{34}{(2)(3)}e^{5t} + \frac{47}{(-11)(-13)}e^{-3t} \\ &= -\frac{7}{11}e^{3t} + \frac{17}{13}e^{5t} + \frac{47}{143}e^{-3t}. \end{aligned}$$

39. Write

$$(s-a_j)\frac{p(s)}{q(s)} = \frac{p(s)}{(q(s)-q(a_j)/(s-a_j))}$$

and take the limit as $s \to a_j$. Finally, use the fact that

$$\lim_{s\to a_j} \frac{q(s)-q(a_j)}{s-a_j} = q'(a_j).$$

3.4 Convolution

1. Let

$$F(s) = \frac{1}{s^2+4} \text{ and } G(s) = \frac{1}{s^2-4}.$$

Then

$$\mathcal{L}^{-1}[F](t) = \frac{1}{2}\sin(2t) \text{ and } \mathcal{L}^{-1}[G](t) = \frac{1}{2}\sinh(2t).$$

By the convolution theorem,

$$\begin{aligned} \mathcal{L}^{-1}[F(s)G(s)](t) &= \frac{1}{2}\sin(2t) * \frac{1}{2}\sinh(2t) \\ &= \frac{1}{4}\int_0^t \sin(2(t-\tau))\sinh(\tau)\,d\tau \\ &= \frac{1}{16}\left[\sin(2(t-\tau))\cosh(2\tau) + \cos(2(t-\tau))\sinh(2\tau)\right]_0^t \\ &= \frac{1}{16}(\sinh(2t) - \sin(2t)). \end{aligned}$$

3. There are two cases. Suppose first that $a^2 \neq b^2$. Then

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\frac{1}{s^2+b^2}\right](t) &= \cos(at) * \frac{1}{b}\sin(bt) \\
&= \frac{1}{b}\int_0^t \cos(a(t-\tau))\sin(b\tau)\,d\tau \\
&= \frac{1}{2b}\int_0^t [\sin((b-a)\tau+at)+\sin((b+a)\tau-at)\,d\tau]\,d\tau \\
&= \frac{1}{2b}\left[-\frac{\cos((b-a)\tau+at)}{b-a}-\frac{\cos((b+a)\tau-at)}{b+a}\right]_0^t \\
&= \frac{1}{2b}\left[-\frac{\cos(bt)}{b-a}-\frac{\cos(bt)}{b+a}+\frac{\cos(at)}{b-a}+\frac{\cos(at)}{b+a}\right] \\
&= \frac{\cos(at)-\cos(bt)}{(b-a)(b+a)}.
\end{aligned}$$

If $a^2 = b^2$, then

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\frac{1}{s^2+b^2}\right](t) &= \cos(at) * \frac{1}{a}\sin(at) \\
&= \frac{1}{a}\int_0^t \cos(a(t-\tau))\sin(a\tau)\,d\tau \\
&= \frac{1}{2a}\int_0^t (\sin(at)+\sin(2a\tau-at))\,d\tau \\
&= \frac{1}{2a}\left[\tau\sin(at)-\frac{1}{2a}\cos(a(2\tau-t))\right]_0^t \\
&= \frac{1}{2a}t\sin(at).
\end{aligned}$$

5. First,

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+a^2)}\right](t) = \frac{1}{a^2}(1-\cos(at)) \text{ and } \mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right](t) = \frac{1}{a}t\sin(at).$$

Then

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{s(s^2+a^2)}\right](t) &= \frac{1}{a^2}[1-\cos(at)] * \sin(at) \\
&= \frac{a}{a^3}\int_0^t [1-\cos(a(t-\tau))]\sin(a\tau)\,d\tau \\
&= \frac{1}{a^3}\left[-\frac{1}{a}\cos(a\tau)-\frac{1}{2}\tau\sin(at)+\frac{1}{4a}\cos(2a\tau-at)\right]_0^t \\
&= \frac{1}{a^4}(1-\cos(at))-\frac{1}{2a^3}\sin(at).
\end{aligned}$$

7.

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{s+2}\frac{e^{-4s}}{s}\right](t) &= e^{-2t} * H(t-4)\\ &= \int_4^t e^{-2(t-\tau)}d\tau\\ &= \begin{cases}\frac{1}{2}e^{-2(t-4)} & \text{if } t > 4,\\ 0 & \text{if } t \le 4.\end{cases}\end{aligned}$$

We can therefore write the inverse transform as

$$\mathcal{L}^{-1}\left[\frac{1}{s+2}\frac{e^{-4s}}{s}\right](t) = \frac{1}{2}(1 - e^{-2(t-4)})H(t-4).$$

9. Take the transform of the initial value problem and solve for $Y(s)$ to get

$$Y(s) = \frac{F(s)}{s^2 - 5s + 6} = \left[\frac{1}{s-3} - \frac{1}{s-2}\right]F(s).$$

By the convolution theorem,

$$y(t) = e^{3t} * f(t) - e^{2t} * f(t).$$

For Problems 11–16 the solution is given, but the details (similar to those of Problems 9 and 10) are omitted.

11.

$$y(t) = \frac{1}{4}e^{6t} * f(t) - \frac{1}{4}e^{2t} * f(t) + 2e^{6t} - 5e^{2t}$$

13.

$$y(t) = \frac{1}{3}\sin(3t) * f(t) - \cos(3t) + \frac{1}{3}\sin(3t)$$

15.

$$y(t) = \frac{1}{4}e^{2t} * f(t) + \frac{1}{12}e^{-2t} * f(t) - \frac{1}{3}e^{t} * f(t) - \frac{1}{4}e^{2t} - \frac{1}{12}e^{-2t} + \frac{4}{3}e^{t}$$

17. The integral equation can be expressed as

$$f(t) = -1 + f(t) * e^{-3t}.$$

Take the transform of this equation to obtain

$$F(s) = -\frac{1}{s} + \frac{F(s)}{s+3}.$$

Then

$$F(s) = -\frac{s+3}{s(s+2)} = \frac{1}{2}\frac{1}{s+2} - \frac{3}{2}\frac{1}{s}.$$

Invert to obtain the solution

$$f(t) = \frac{1}{2}e^{-2t} - \frac{3}{2}.$$

19. The equation is $f(t) = e^{-t} + f(t) * 1$. Take the transform and solve for $F(s)$ to get

$$F(s) = \frac{s}{(s-1)(s+1)} = \frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s-1}.$$

Then

$$f(t) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{t} = \cosh(t).$$

21. The equation is $f(t) = 3 + f(t) * \cos(2t)$. From this we obtain

$$F(t) = \frac{3(s^2+4)}{s(s^2-s+4)} = \frac{3}{s} + \frac{3}{s^2-s+4}.$$

The inverse of this is

$$f(t) = 3 + \frac{2\sqrt{15}}{5}e^{t/2}\sin\left(\frac{\sqrt{15}}{2}t\right).$$

23. Let $F = \mathcal{L}[f]$ and $G = \mathcal{L}[g]$. Then

$$F(s)G(s) = F(s)\int_0^\infty e^{-s\tau}g(\tau)\,d\tau.$$

Now recall that

$$e^{-s\tau}F(s) = \mathcal{L}[H(t-\tau)f(t-\tau)](s).$$

Substitute this into the expression for $F(s)G(s)$ to get

$$F(s)G(s) = \int_0^\infty \mathcal{L}[H(t-\tau)f(t-\tau)](s)g(\tau)\,d\tau.$$

But, from the definition of the Laplace transform,

$$\mathcal{L}[H(t-\tau)f(t-\tau)] = \int_0^\infty e^{-st}H(t-\tau)f(t-\tau)\,dt.$$

Then

$$\begin{aligned} F(s)G(s) &= \int_0^\infty \left[\int_0^\infty e^{-st}H(t-\tau)f(t-\tau)\,dt\right]g(\tau)\,d\tau \\ &= \int_0^\infty \int_0^\infty e^{-st}g(\tau)H(t-\tau)f(t-\tau)\,d\tau\,dt. \end{aligned}$$

But $H(t-\tau) = 0$ if $0 \le t < \tau$, while $H(t-\tau) = 1$ if $t \ge \tau$. Therefore

$$F(s)G(s) = \int_0^\infty \int_\tau^\infty e^{-st}g(\tau)f(t-\tau)\,dt\,d\tau.$$

The last integration is over the wedge in the $t, \tau-$ plane consisting of points (t, τ) with $0 \leq \tau \leq t < \infty$. Reverse the order of integration to write

$$\begin{aligned} F(s)G(s) &= \int_0^\infty \int_0^t e^{-st} g(\tau) f(t-\tau)\, d\tau\, dt \\ &= \int_0^\infty e^{-st} \left[\int_0^t g(\tau) f(t-\tau)\, d\tau \right] dt \\ &= \int_0^\infty e^{-st} (f * g)(t)\, dt \\ &= \mathcal{L}[f * g](s). \end{aligned}$$

This is what we wanted to show.

3.5 Impulses and the Dirac Delta Function

In Problem 1 details of the solution are given. For Problems 2–5, the details are similar and only the solution is given.

1. Transform the initial value problem to obtain

$$(s^2 + 5s + 6)Y(s) = 3e^{-2s} - 4e^{-5s}.$$

Using a partial fractions decomposition, this gives us

$$Y(s) = 3\left[\frac{1}{s+2} - \frac{1}{s+3}\right] e^{-3s} - 4\left[\frac{1}{s+2} - \frac{1}{s+3}\right] e^{-5s}.$$

Invert this to obtain the solution

$$y(t) = 3\left[e^{-2(t-2)} - e^{-3(t-2)}\right] H(t-2) - 4\left[e^{-2(t-5)} - e^{-3(t-5)}\right] H(t-5).$$

3.

$$y(t) = 6(e^{-2t} - e^{-t} + te^{-t})$$

5.

$$y(t) = (B+9)e^{-2t} - (B+6)e^{-3t}$$

3.6 Systems of Linear Differential Equations

1. Take the transform of the system:

$$sX - 2sY = \frac{1}{s}, sX - X + Y = 0.$$

Then

$$
\begin{aligned}
X(s) &= \frac{1}{s^2(2s-1)} = -\frac{1}{s^2} - \frac{2}{s} + \frac{4}{2s-1}, \\
Y(s) &= \frac{1-s}{s^2(2s-1)} = -\frac{1}{s^2} - \frac{1}{s} + \frac{2}{2s-1}.
\end{aligned}
$$

Apply the inverse transform to get the solution

$$x(t) = -t - 2 + 2e^{t/2}, y(t) = -t - 1 + e^{t/2}.$$

3. After transforming the system, we obtain

$$sX + (2s-1)Y = \frac{1}{s}, 2sX + Y = 0.$$

Then

$$
\begin{aligned}
X(s) &= -\frac{1}{s^2(4s-3)} = \frac{4}{9s} - \frac{16}{9(4s-3)} + \frac{1}{3s^2}, \\
Y(s) &= \frac{2}{s(4s-3)} = -\frac{2}{3s} + \frac{8}{3(4s-3)}.
\end{aligned}
$$

Invert these to obtain

$$
\begin{aligned}
x(t) &= \frac{4}{9}(1 - e^{3t/4}) + \frac{1}{3}t, \\
y(t) &= \frac{2}{3}(-1 + e^{3t/4}).
\end{aligned}
$$

5. The system transforms to

$$3sX - Y = \frac{2}{s^2}sX + sY - Y = 0.$$

Then

$$
\begin{aligned}
X(s) &= \frac{2(s-1)}{s^2(3s-2)} = \frac{3}{4s} + \frac{1}{2s^2} + \frac{1}{s^3} - \frac{9}{4(3s-2)}, \\
Y(s) &= -\frac{2}{s^2(3s-2)} = \frac{3}{2s} + \frac{1}{s^2} - 9\frac{2(3s-2)}{.}
\end{aligned}
$$

Then

$$
\begin{aligned}
x(t) &= \frac{3}{4} + \frac{1}{2}t + \frac{1}{2}t^2 - \frac{3}{4}e^{2t/3}, \\
y(t) &= \frac{3}{2} + t - \frac{3}{2}e^{2t/3}.
\end{aligned}
$$

7. The transform of the system is

$$sX + 2X - sY = 0, sX + Y + X = \frac{2}{s^3}.$$

Then

$$X(s) = \frac{2}{s^2(s^2+2s+2)} = \frac{1}{s^2} + \frac{s+1}{s^2+2s+2} - \frac{1}{s},$$
$$Y(s) = \frac{2(s+2)}{s^3(s^2+2s+2)} = -\frac{1}{s^2} + \frac{1}{s^2+2s+2} + \frac{2}{s^3}.$$

The solution is

$$x(t) = t + e^{-t}\cos(t) - 1$$
$$y(t) = -t + e^{-t}\sin(t) + t^2.$$

In inverting $X(s)$ and $Y(s)$, the terms involving s^2+2s+2 can be treated by using a shifting theorem, expressing these as functions of $s+1$.

9. First,

$$sX + sY + X - Y = 0, sX + 2sY + X = \frac{1}{s}.$$

Then

$$X(s) = \frac{1-s}{s(s+1)^2} = \frac{-2}{s+1)^2} - \frac{1}{s+1} + \frac{1}{s},$$
$$Y(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}.$$

Then

$$x(t) = 1 - e^{-t}(2t+1), y(t) = 1 - e^{-t}.$$

11. The system transforms to

$$sX - 2sY + 3X = 0, X - 4sY + 3sZ = \frac{1}{s^2}, X - 2sY + 3sZ = -\frac{1}{s}.$$

Then

$$X(s) = \frac{s+1}{s^2(s+3)} = \frac{2}{9s} - \frac{2}{9(s+3)} - \frac{1}{3s^2},$$
$$Y(s) = -\frac{1}{2}\frac{s+1}{s^3} = \frac{-1}{2s^3} - \frac{1}{2s^2},$$
$$Z(s) = -\frac{2}{3}\frac{s^2+3s+1}{s^3(s+3)} = -\frac{2}{9s} - \frac{2}{81s} + \frac{2}{81(s+3)} - \frac{16}{27s^2}.$$

The solution is

$$x(t) = \frac{2}{9} + \frac{1}{3}t - \frac{2}{9}e^{-3t},$$
$$y(t) = -\frac{1}{4}t(t+2)$$
$$z(t) = -\frac{16}{27}t - \frac{1}{9}t^2 - \frac{2}{81} + \frac{2}{81}e^{-3t}.$$

13. The equations for the loop currents are

$$20i_1' + 10(i_1 - i_2) = E(t) = 5H(t-5),$$
$$30i_2' + 10i_2 + 10(i_2 - i_1) = 0.$$

Initial conditions are

$$i_1(0) = i_2(0) = 0.$$

Transform the system to obtain

$$\begin{aligned} I_1(s) &= \frac{5(30s+20)e^{-5s}}{s(600s^2+700s+100)} \\ &= \left[\frac{1}{s} - \frac{1}{10(s+1)} - \frac{27}{5}\frac{1}{6s+1}\right]e^{-5s}, \\ I_2(s) &= \frac{50e^{-5s}}{s(600s^2+700s+100)} \\ &= \left[\frac{1}{2s} + \frac{10}{s+1} - \frac{18}{5}\frac{1}{6s+1}\right]e^{-5s}. \end{aligned}$$

Invert these to obtain the current functions:

$$\begin{aligned} i_1(t) &= \left[1 - \frac{1}{10}e^{-(t-5)} - \frac{9}{10}e^{-(t-5)/6}\right]H(t-5), \\ i_2(t) &= \left[\frac{1}{2} + \frac{1}{10}e^{-(t-5)} - \frac{3}{10}e^{-(t-5)/6}\right]H(t-5). \end{aligned}$$

15. Using the notation of the preceding problem, we can write

$$\begin{aligned} x_1' &= -\frac{6}{200}x_1 + \frac{3}{100}x_2, \\ x_2' &= \frac{4}{200}x_1 - \frac{4}{200}x_2 + 5H(t-3). \end{aligned}$$

Initial conditions are $x_1(0) = 10, x_2(0) = 5$. Apply the transform to this initial value problem and rearrange terms to obtain

$$(100s+3)X_1 - 3X_2 = 1000,$$
$$-2X_1 + (100s+4)X_2 = 500 + 500e^{-3s}.$$

Solve these to get

$$\begin{aligned} X_1(s) &= \frac{100000s + 5500 + 1500e^{-3s}}{10000s^2 + 700s + 6} \\ &= \frac{50}{50s+3} + \frac{900}{100s+1} + \left[\frac{300}{100s+1} - \frac{150}{50s+3}\right]e^{-3s}, \end{aligned}$$

and

$$\begin{aligned} X_2(s) &= \frac{50000s + 3500 + (50000s + 1500)e^{-3s}}{10000s^2 + 700s + 6} \\ &= -\frac{50}{50s+3} + \frac{600}{100s+1} + \left[\frac{150}{50s+3} + \frac{200}{100s+1}\right] e^{-3s}. \end{aligned}$$

Apply the inverse transform to obtain the solution:

$$\begin{aligned} i_1(t) &= e^{-3t/50} + 9e^{-t/100} + 3(e^{-(t-3)/100} - e^{-3(t-3)/50})H(t-3), \\ i_2(t) &= -e^{-3t/50} + 6e^{-t/100} + (3e^{-3(t-3)/50} + 2e^{-(t-3)/100})H(t-3). \end{aligned}$$

Chapter 4

Sturm-Liouville Problems and Eigenfunction Expansions

4.1 Eigenvalues, Eigenfunctions and Sturm-Liouville Problems

For these problems, an eigenfunction is found for each eigenvalue, and it is understood that nonzero constant multiples of eigenfunctions are also eigenfunctions.

1. The problem is regular on $[0, L]$. To find the eigenvalues and eigenfunctions, take cases on λ.

   Case 1. If $\lambda = 0$, the differential equation is $y'' = 0$, with solutions $y = a + bx$. Now $y(0) = a = 0$, so $y = bx$. But then $y'(L) = b = 0$ also, so this case has only the trivial solution and 0 is not an eigenvalue of this problem.

   Case 2. If λ is negative, say $\lambda = -\alpha^2$ with $\alpha > 0$, then the differential equation is
   $$y'' - \alpha^2 y = 0$$
   with general solution
   $$y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}.$$
   Now
   $$y(0) = c_1 + c_2 = 0,$$
   so $c_2 = -c_1$ and
   $$y(x) = c_1 \left(e^{\alpha x} - e^{-\alpha x}\right) = 2c_1 \sinh(\alpha x).$$

From the other boundary condition,

$$y'(L) = 2c_1\alpha\cosh(\alpha L) = 0.$$

But $\cosh(\alpha L) > 0$ and $\alpha > 0$, so we must have $c_1 = 0$ and this case also has only the trivial solution. This problem has no positive eigenvalue.

Case 3. Suppose λ is positive, write $\lambda = \alpha^2$, with $\alpha > 0$. Now the differential equation is

$$y'' + \alpha^2 y = 0,$$

with general solution

$$y(x) = c_1\cos(\alpha x) + c_2\sin(\alpha x).$$

Immediately $y(0) = c_1 = 0$, so

$$y(x) = c_2\sin(\alpha x).$$

From the other boundary condition, we must have

$$y'(L) = c_2\alpha\cos(\alpha L) = 0.$$

We need to be able to choose $c_2 \neq 0$ to have nontrivial solutions. This requires that we α must be chosen to satisfy

$$\cos(\alpha L) = 0.$$

We know that the zeros of the cosine function have the form $(2n-1)\pi/2$ for integer π, so let

$$\alpha L = \frac{(2n-1)\pi}{2},$$

with $n = 1, 2, \cdots$. Then acceptable values of α are

$$\alpha = \frac{(2n-1)\pi}{2L}.$$

Because $\lambda = \alpha^2$, the eigenvalues of this problem, indexed by n, are

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2$$

for $n = 1, 2, \cdots$. Corresponding eigenfunctions are

$$\varphi_n(x) = \sin\left(\frac{(2n-1)\pi}{2L}x\right),$$

or any nonzero constant multiple of this function.

3. The problem is regular on $[0, L]$;

$$\lambda_n = \left[\left(n - \frac{1}{2}\right)\frac{\pi}{4}\right]^2$$

is an eigenvalue for $n = 1, 2, \cdots$, with eigenfunctions

$$\varphi_n(x) = \cos\left(\frac{(2n-1)\pi}{8}\right).$$

5. The problem is periodic on $[-3\pi, 3\pi]$. Eigenvalues are

$$\lambda_0 = 0 \text{ and } \lambda_n = \frac{n^2}{9} \text{ for } n = 1, 2, \cdots .$$

Eigenfunctions are

$$\varphi_n(x) = a_n \cos(nx/3) + b_n \sin(nx/3)$$

for $n = 0, 1, 2, \cdots$, with a_n and b_n constant and not both zero.

7. The problem is regular on $[0, 1]$. The analysis to find eigenvalues and eigenfunctions is similar to that done for Problem 6. Take cases on λ. It is routine to check that $\lambda = 0$ or $\lambda < 0$ lead only to the trivial solution, so the problem has no negative eigenvalue and zero is not an eigenvalue. If $\lambda = \alpha^2$ for $\alpha > 0$, then

$$y(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x).$$

Using the first boundary condition,

$$y(0) - 2y'(0) = 0 = c_1 - 2c_2\alpha$$

so $c_1 = 2\alpha c_2$ and

$$y(x) = 2\alpha c_2 \cos(\alpha x) + c_2 \sin(\alpha x).$$

Now use the boundary condition at 1:

$$y'(1) = -2\alpha^2 c_2 \sin(\alpha) + c_2 \alpha \cos(\alpha) = 0.$$

For a nontrivial solution we need to be able to choose c_2 nonzero. This requires that

$$-2\alpha \sin(\alpha) + \cos(\alpha) = 0,$$

or

$$\tan(\alpha) = \frac{1}{2\alpha}.$$

Solutions of this equation must be numerically approximated. There are infinitely many positive solutions $\alpha_1 < \alpha_2 < \cdots$, and the eigenvalues are $\lambda_j = \alpha_j^2$. The first four eigenvalues are

$$\lambda_1 \approx 0.42676, \lambda_2 \approx 10.8393, \lambda_3 \approx 40.4702, \lambda_4 \approx 89.8227.$$

Corresponding eigenfunctions are

$$\varphi_n(x) = 2\sqrt{\lambda}\cos(\sqrt{\lambda_n}x) + \sin(\sqrt{\lambda_n}x).$$

9. The problem is regular on $[0, \pi]$. The differential equation can be written

$$y'' + 2y' + \lambda y = 0.$$

and the characteristic equation has roots

$$-1 \pm \sqrt{1-\lambda}.$$

Here it is convenient to take cases on $1 - \lambda$.

Case 1. If $1 - \lambda = 0$, then $\lambda = 1$ and the general solution is

$$y(x) = c_1 e^{-x} + c_2 x e^{-x}.$$

Now $y(0) = c_1 = 0$, so $y(x) = c_2 x e^{-x}$. And

$$y(\pi) = 0 = c_2 \pi e^{-\pi} = 0$$

forces $c_2 = 0$, so this case has only the trivial solution and 0 is not an eigenvalue.

Case 2. If $1 - \lambda$ is positive, say $1 - \lambda = \alpha^2$ with $\alpha > 0$, then

$$y(x) = c_1 e^{(-1+\alpha)x} + c_2 e^{(-1-\alpha)x}.$$

Now

$$y(0) = c_1 + c_2 = 0$$

so $c_2 = -c_1$ and

$$y(x) = c_1\left(e^{(-1+\alpha)x} - e^{(-1-\alpha)x}\right).$$

Then

$$y(\pi) = c_1\left(e^{(-1+\alpha)\pi} - e^{(-1-\alpha)\pi}\right).$$

If $c_1 \neq 0$, this requires that

$$e^{\alpha\pi} = e^{-\alpha\pi},$$

which is impossible if $\alpha > 0$. The problem has no negative eigenvalue.

Case 3. If $1 - \lambda$ is negative, write $1 - \lambda = -\alpha^2$. Now

$$y(x) = c_1 e^{-x}\cos(\alpha x) + c_2 e^{-x}\sin(\alpha x).$$

Immediately $y(0) = c_1 = 0$. Next,

$$y(\pi) = c_2 e^{-\pi}\sin(\alpha\pi) = 0.$$

To have $c_2 \neq 0$, we must choose

$$\alpha = \sqrt{\lambda - 1} = n,$$

any positive integer. Then $\lambda = 1 + n^2$, so the eigenvalues are

$$\lambda_n = 1 + n^2 \text{ for } n = 1, 2, \cdots .$$

Eigenfunctions are

$$\varphi_n(x) = e^{-x} \sin(nx).$$

4.2 Eigenfunction Expansions

In Problems 1–5, the weight function is $p(x) = 1$ (read from the differential equation). In Problem 6 the differential equation must be put into standard Sturm-Liouville form to read the weight function $p(x)$ as the coefficient of λ.

In graphing partial sums of eigenfunctions and comparing them to the function, note the differences in the number of terms that must be taken to have the partial sum fit reasonably close to the function. The convergence theorem does not give any information about how fast an eigenfunction expansion converges to the function.

1. It is routine to find the eigenfunctions

$$\varphi_n(x) = \sin\left(\frac{n\pi x}{2}\right)$$

for this problem. The expansion has the form

$$\sum_{n=1}^{\infty} c_n \sin(n\pi x/2),$$

where

$$c_n = \frac{\int_0^2 (1-\xi)\sin(n\pi\xi/2)\,d\xi}{\int_0^2 \sin^2(n\pi\xi/2)\,d\xi}.$$

These integrals are

$$\int_0^2 \sin^2(n\pi\xi/2)\,d\xi = 1$$

and

$$\int_0^2 (1-\xi)\sin(n\pi\xi/2)\,d\xi = \frac{2(1+(-1)^n)}{n\pi}.$$

The eigenfunction expansion on $[0, 2]$ is

$$\sum_{n=1}^{\infty} \frac{2(1+(-1)^n)}{n\pi} \sin(n\pi x/2).$$

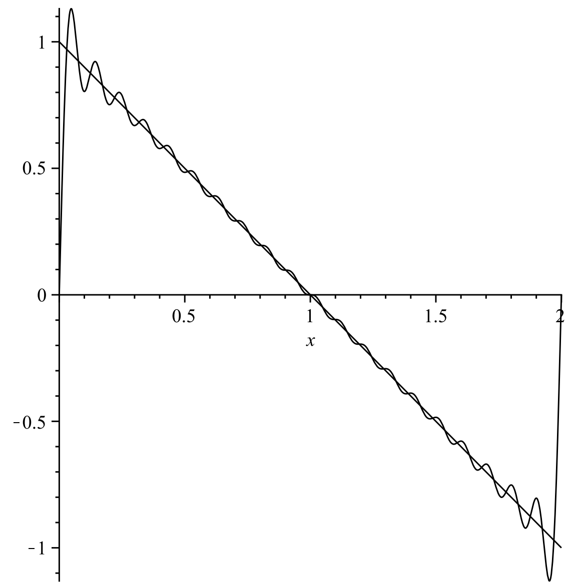

Figure 4.1: Comparison of $1-x$ and the fortieth partial sum of its eigenfunction expansion on $[0,2]$.

Figure 4.1 shows a graph of $f(x) = 1 - x$ and the fortieth partial sum of this expansion. By the convergence theorem, this expansion converges to $1 - x$ for $0 < x < 2$. Clearly the expansion converges to 0 at both $x = 0$ and $x = 2$ because the eigenfunctions vanish there.

3. The eigenfunctions are

$$\varphi_n(x) = \cos((2n-1)\pi x/8).$$

The coefficients in the expansion of $f(x)$ on $[0,4]$ are

$$\begin{aligned} c_n &= \frac{\int_0^2 -\cos((2n-1)\pi x/8)\,dx + \int_2^4 \cos((2n-1)\pi x/8)\,dx}{\int_0^4 \cos^2((2n-1)\pi x/8)\,dx} \\ &= \frac{4}{(2n-1)\pi}\left[(-1)^{n+1} + \sqrt{2}(\cos(n\pi/2) - \sin(n\pi/2))\right]. \end{aligned}$$

The expansion has the form

$$\sum_{n=1}^{\infty} c_n \cos((2n-1)\pi x/8).$$

Figure 4.2 compares $f(x)$ with the sixtieth partial sum of this eigenfunction expansion. The theorem tells us that the expansion converges to $f(x)$

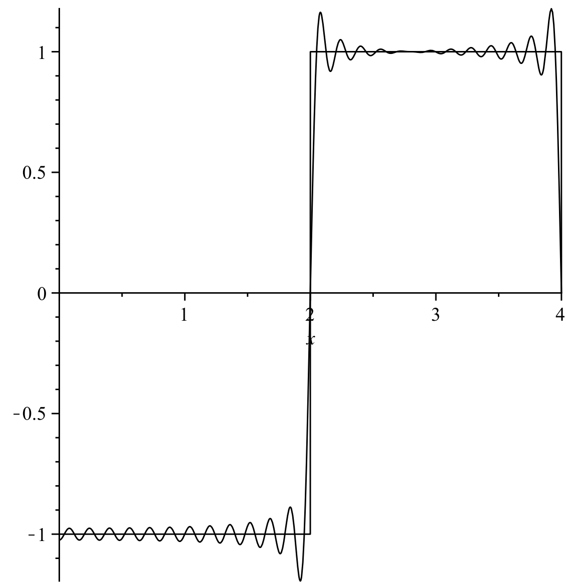

Figure 4.2: Comparison of $f(x)$ and the sixtieth partial sum of its eigenfunction expansion on $[0, 4]$.

on $(0, 2)$ and on$(2, 4)$, as well at to 0 at $x = 0$ (average of left and right limits there).

5. The eigenfunctions are

$$\varphi_0(x) = 1, \varphi_n(x) = a_n \cos(nx/3) + b_n \sin(nx/3)$$

for $n = 1, 2, \cdots$. The coefficients in the eigenfunction expansion of x^2 on $[-3\pi, 3\pi]$ are

$$\begin{aligned} a_0 &= \frac{1}{6\pi} \int_{-3\pi}^{3\pi} \xi^2 \, d\xi = 3\pi^2, \\ a_n &= \frac{1}{3\pi} \xi^2 \cos(n\xi/3) \, d\xi = \frac{36}{n^2}(-1)^n, \\ b_n &= \frac{1}{3\pi} \int_{-3\pi}^{3\pi} \xi^2 \sin(n\xi/3) \, d\xi = 0. \end{aligned}$$

The eigenfunction expansion is

$$3\pi^2 + 36 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx/3).$$

Figure 4.3 shows $f(x)$ and the fifth partial sum of this eigenfunction expansion. By the theorem, the expansion converges to x^2 for $-3\pi < x < 3\pi$.

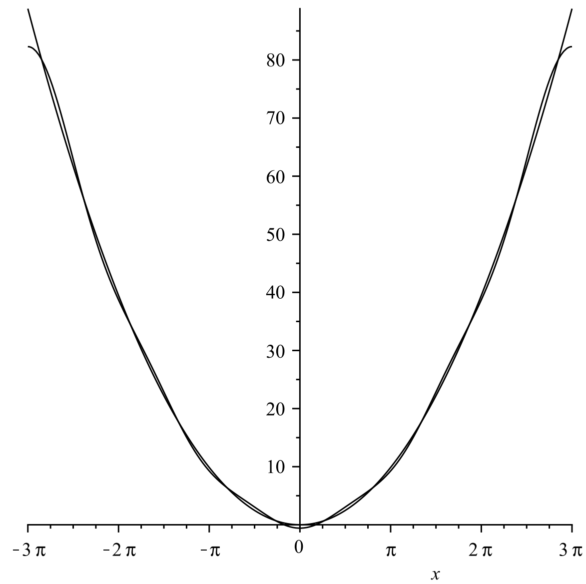

Figure 4.3: Comparison of $f(x)$ and the fifth partial sum of its eigenfunction expansion in Problem 5.

7. Recall that the complex conjugate of $z = a + ib$ is $\overline{z} = a - ib$. Suppose λ is an eigenvalue of a Sturm-Liouville problem, with eigenfunction $\varphi(x)$. By taking the complex conjugate of the Sturm-Liouville differential equation and appropriate boundary conditions, it is routine to check that $\overline{\lambda}$ is also an eigenvalue with eigenfunction $\overline{\varphi}(x)$. If λ is complex and not real, then $\lambda \neq \overline{\lambda}$, so the eigenfunctions must be orthogonal with respect to the weight function p, and

$$\int_a^b p(x)\varphi(x)\overline{\varphi}(x)\,dx = 0.$$

Now,

$$\varphi(x)\overline{\varphi}(x) = |\varphi(x)|^2,$$

so

$$\int_a^b p(x)|\varphi(x)|^2|\,dx = 0.$$

This is impossible because $p(x) > 0$ on (a, b) and $\varphi(x)$ is continuous and not identically zero on the interval. This contradiction shows that $\lambda = \overline{\lambda}$, so λ is real.

4.3 Fourier Series

1. The Fourier series of $f(x) = 4$ on $[-3, 3]$ has the form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n \cos(n\pi x/3) + b_n \sin(n\pi x/3)].$$

All that is left is to compute the coefficients. First, because $f(x)$ is an even function, each $b_n = 0$. Compute

$$a_0 = \frac{2}{3}\int_{-3}^{3} 4\,dx = 8$$

and, for $n = 1, 2, \cdots$,

$$a_n = \frac{2}{3}\int_{0}^{3} 4\cos(n\pi x/3)\,d\xi = 0.$$

With each $a_n = 0$ for $n = 1, 2, \cdots$, the Fourier series is

$$\frac{1}{2}a_0 = 4.$$

This series consists of a single term, namely the constant term (which seems obvious by hindsight, if not noticed immediately). This one-term series converges to 4 on $[-3, 3]$.

3. Because $\cosh(\pi x)$ is an even function, each $b_n = 0$ in the Fourier series, which will have the appearance

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x).$$

Compute

$$a_0 = 2\int_{0}^{1} \cosh(\pi x)\,dx \frac{2}{\pi}\sinh(\pi)$$

and, for $n = 1, 2, \cdots$,

$$a_n = 2\int_{0}^{1} \cosh(\pi x)\cos(n\pi x)\,dx = \frac{2\sinh(\pi)}{\pi}\frac{(-1)^n}{1+n^2}.$$

The Fourier series is

$$\frac{1}{\pi}\sinh(\pi) + \sum_{n=1}^{\infty}\frac{2\sinh(\pi)}{\pi}\frac{(-1)^n}{1+n^2}\cos(n\pi x).$$

This series converges to $\cosh(\pi x)$ for $-1 \leq x \leq 1$. Figure 4.4 shows a graph of $f(x)$ and the eighth partial sum of this Fourier series.

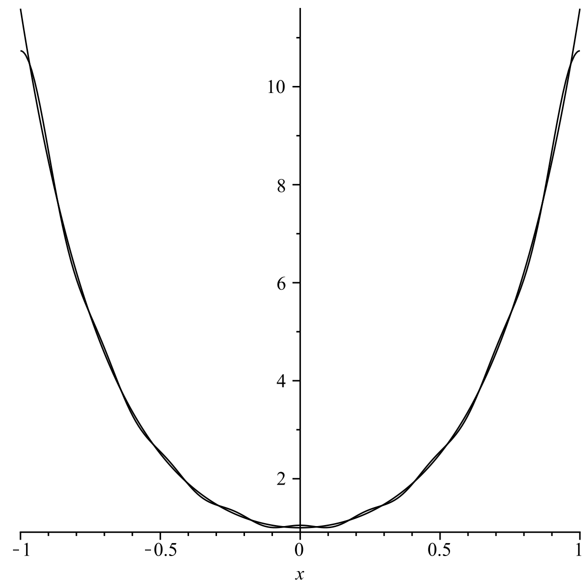

Figure 4.4: Comparison of $f(x)$ and the eighth partial sum of the Fourier series in Problem 3.

5. The Fourier series of $f(x)$ on $[-\pi, \pi]$ is

$$\frac{16}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\sin((2n-1)x).$$

This series converges to

$$\begin{cases} -4 & \text{for } -\pi < x < 0, \\ 4 & \text{for } 0 < x < \pi, \\ 0 & \text{for } x = 0, -\pi, \pi. \end{cases}$$

Figure 4.5 is a graph of $f(x)$ and the twentieth partial sum of this Fourier series.

7. The Fourier series of $f(x)$ on $[-2, 2]$ is

$$\frac{13}{3}+\sum_{n=1}^{\infty}(-1)^n\left[\frac{16}{(n\pi)^2}\cos(n\pi x/2)+\frac{1}{n\pi}\sin(n\pi x/2)\right].$$

This series converges to

$$\begin{cases} x^2-x+3 & \text{for } -2 < x < 2, \\ 7 & \text{for } x = \pm 2. \end{cases}$$

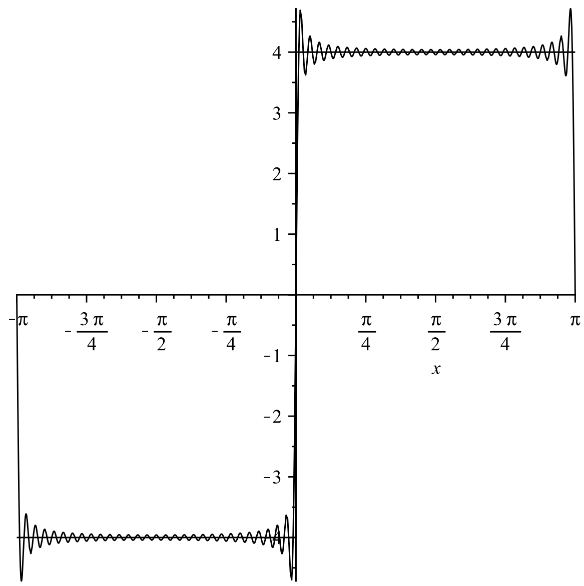

Figure 4.5: Comparison of $f(x)$ and the twentieth partial sum of the Fourier series in Problem 5.

At the endpoints,

$$\frac{1}{2}(f(-2+)+f(2-))=\frac{1}{2}(9+5)=7.$$

Figure 4.6 shows $f(x)$ and the twentieth partial sum of this Fourier series.

9. The Fourier series of $f(x)$ on $[-\pi,\pi]$ is

$$\frac{3}{2}+\frac{2}{\pi}\sum_{n-1}^{\infty}\frac{1}{2n-1}\sin((2n-1)x).$$

This series converges to

$$\begin{cases} 1 & \text{for } -\pi<x<0, \\ 2 & \text{for } 0<x<\pi, \\ 3/2 & \text{for } x=0,-\pi,\pi, \end{cases}$$

Figure 4.7 shows the function and the thirtieth partial sum of the Fourier series in Problem 9.

11. The Fourier series of $\cos(x)$ on $[-3,3]$ is

$$\frac{1}{3}\sin(3)+6\sin(3)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\pi^2n^2-9}\cos(n\pi x/3).$$

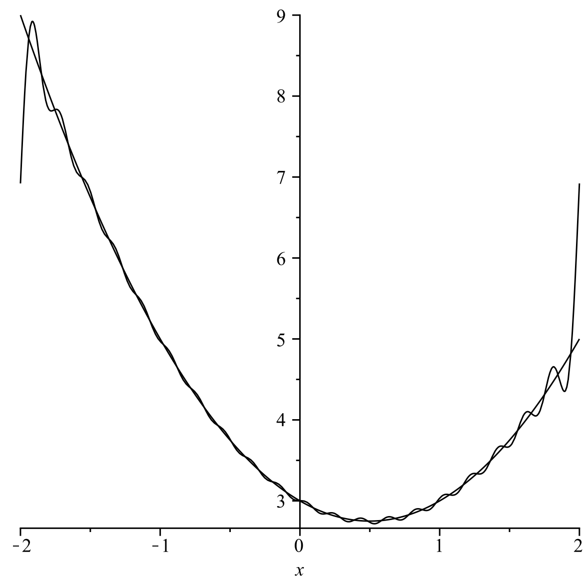

Figure 4.6: Comparison of $f(x)$ and the twentieth partial sum of the Fourier series in Problem 7.

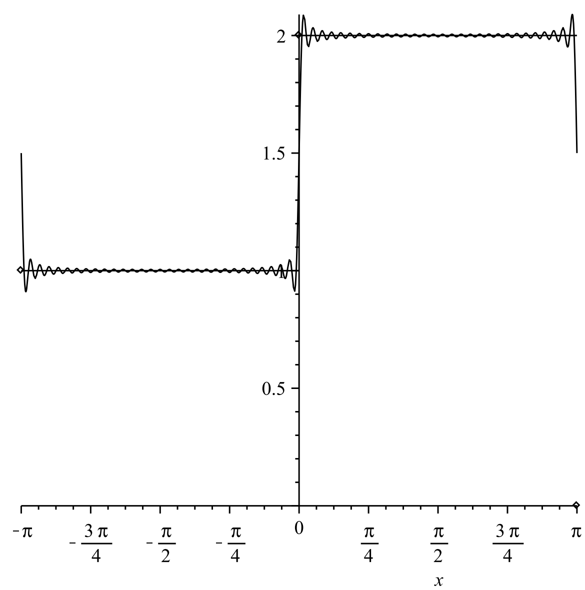

Figure 4.7: Comparison of $f(x)$ and the thirtieth partial sum of the Fourier series in Problem 9.

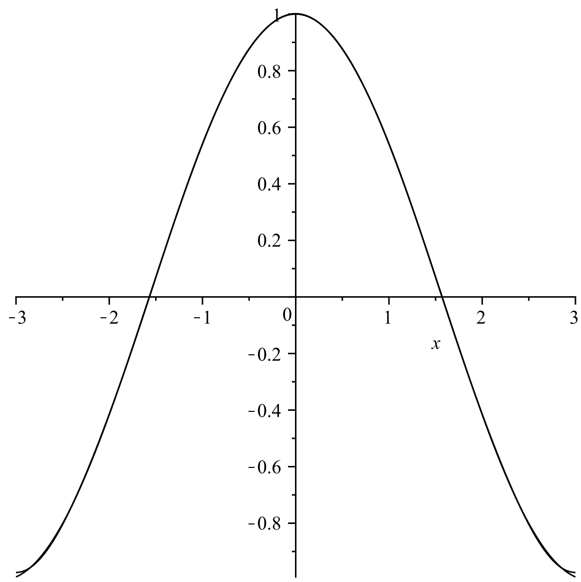

Figure 4.8: Comparison of $f(x)$ and the fifth partial sum of the Fourier series in Problem 11.

This converges to $\cos(x)$ on $[-3, 3]$. Figure 4.8 is a graph of $f(x)$ and the fifth partial sum of this Fourier expansion on $[-3, 3]$.

It might seem at first that $\cos(x)$ should be its own Fourier expansion, but this problem illustrates the importance of the interval. If you expand $\cos(x)$ in a Fourier series on $[-\pi, \pi]$, you obtain just $\cos(x)$. But this is not the expansion on $[-3, 3]$.

13. The Fourier series of $f(x)$ on $[-3, 3]$ converges to

$$\begin{cases} 3/2 & \text{for } x = \pm 3, \\ 2x & \text{for } -3 < x < -2, \\ -2 & \text{for } x = -2, \\ 0 & \text{for } -2 < x < 1, \\ 1/2 & \text{for } x = 1, \\ x^2 & \text{for } 1 < x < 3. \end{cases}$$

15. The Fourier series converges to

$$\begin{cases} -1 & \text{for } x = \pm 4, \\ 3/2 & \text{for } x = -2, \\ 5/2 & \text{for } x = 2, \\ f(x) & \text{for all other } x \text{ in } [-4, 4]. \end{cases}$$

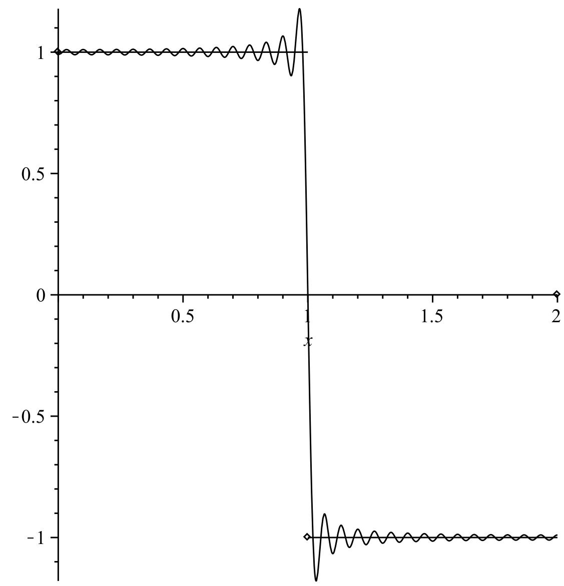

Figure 4.9: Comparison of $f(x)$ and the thirtieth partial sum of the Fourier cosine series in Problem 21.

17. The argument is like that used in Problem 16, except now use the fact that $f(-x) = -f(x)$.

19. Suppose $f(x)$ is both even and odd on $[-L, L]$. Then, for every x in this interval,

$$f(x) = f(-x) = -f(x).$$

But then $f(x) = 0$, so $f(x)$ is identically zero on the interval.

21. The cosine series is

$$-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\cos((2n-1)\pi x/2).$$

This converges to

$$\begin{cases} 1 & \text{for } 0 \leq x < 1, \\ 0 & \text{for } x = 1, \\ -1 & \text{for } 1 < x \leq 2. \end{cases}$$

Figure 4.9 compares the function to the thirtieth partial sum of this cosine series.

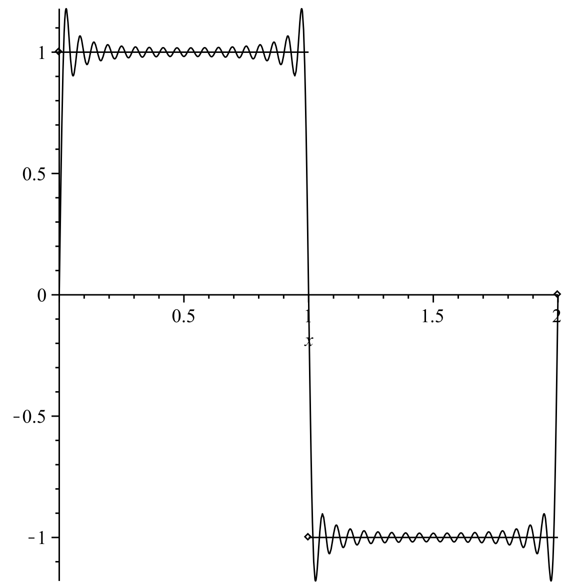

Figure 4.10: Comparison of $f(x)$ and the seventieth partial sum of the Fourier sine series in Problem 21.

The sine series is

$$\frac{2}{\pi}\sum_{n=1}^{\infty}(1+(-1)^n-2\cos(n\pi/2))\sin(n\pi x/2),$$

which converges to

$$\begin{cases} 1 & \text{for } 0<x<1, \\ 0 & \text{for } x=0,1,2, \\ -1 & \text{for } 1<x<2. \end{cases}$$

Figure 4.10 shows $f(x)$ and the seventieth partial sum of this sine expansion on $[0,2]$.

23. The cosine series is

$$1-\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos((2n-1)\pi x),$$

converging to $2x$ for $0 \le x \le 1$.

The sine series is

$$-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\pi x),$$

converging to $2x$ for $0<x<1$ and to 0 for $x=0$ and for $x=1$.

25. The cosine series is

$$-1 - e^{-1} + 2\sum_{n=1}^{\infty} \frac{1-(-1)^n e^{-1}}{1+n^2\pi^2} \cos(n\pi x),$$

converging to e^{-x} for $0 \le x \le 1$.

The sine series is

$$2\pi \sum_{n=1}^{\infty} \left[\frac{n}{1+n^2\pi^2}(1-(-1)^n e^{-1}) \right] \sin(n\pi x),$$

converging to e^{-x} for $0 < x < 1$ and to 0 at $x = 0$ and at $x = 1$.

27. The cosine expansion is

$$-\frac{1}{5} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \cos(n\pi/5)\sin(2n\pi/5)\cos(n\pi x/5),$$

converging to

$$\begin{cases} 1 & \text{for } 0 \le x < 1, \\ 1/2 & \text{for } x = 1, \\ 0 & \text{for } 1 < x < 3, \\ -1/2 & \text{for } x = 3, \\ -1 & \text{for } 3 < x < 5. \end{cases}$$

Figure 4.11 shows the function and the sixtieth partial sum of this cosine expansion.

The sine series is

$$\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n}(1+(-1)^n - 2\cos(n\pi/5)\cos(2n\pi/5))\sin(n\pi x/5),$$

converging to

$$\begin{cases} 1 & \text{for } 0 < x < 1, \\ 1/2 & \text{for } x = 1, \\ 0 & \text{for } 1 < x < 3,\ x = 0, \text{ or } x = 5, \\ -1/2 & \text{for } x = 3, \\ -1 & \text{for } 3 < x < 5. \end{cases}$$

Figure 4.12 shows $f(x)$ and the one hundredth partial sum of its sine expansion on $[0, 5]$.

29. The cosine series is

$$-1 - \frac{24}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[2(-1)^n + \frac{4}{n^2\pi^2}(1-(-1)^n) \right] \cos(n\pi x/3),$$

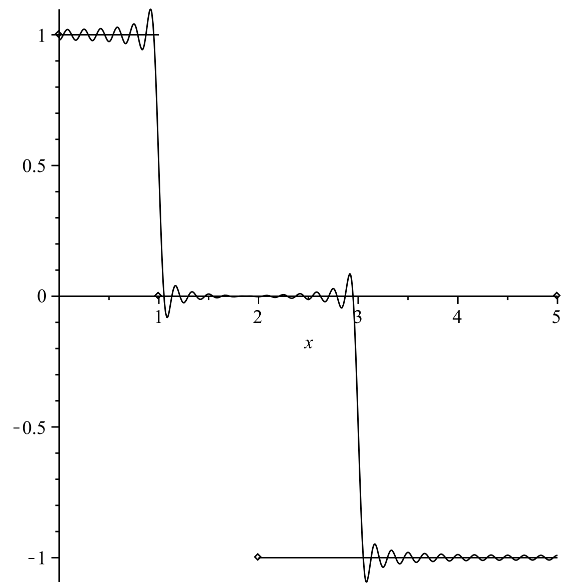

Figure 4.11: $f(x)$ and the sixtieth partial sum of the cosine series in Problem 27.

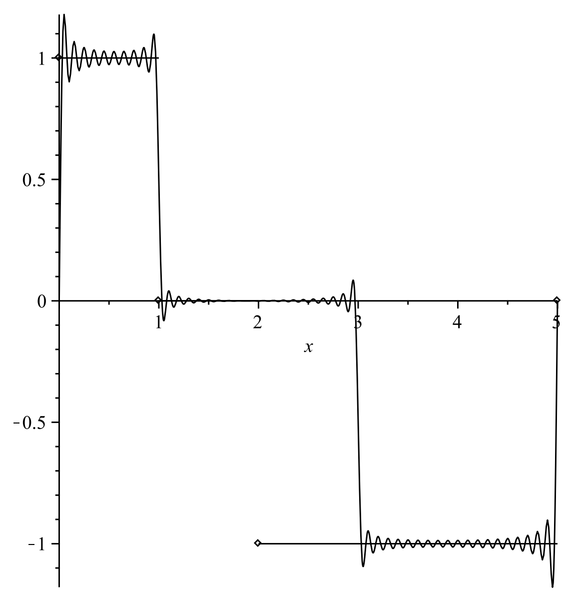

Figure 4.12: Comparison of $f(x)$ and the hundredth partial sum of the sine series in Problem 27.

converging to $1 - x^2$ for $0 \leq x \leq 2$.

The sine series is

$$\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[1 + 7(-1)^n - \frac{48}{n^2\pi^2} \right] \sin(n\pi x/2).$$

This series converges to $1 - x^2$ for $0 < x < 2$ and to 0 at $x = 0$ and $x = 2$.

Chapter 5

The Heat Equation

5.1 Diffusion Problems on a Bounded Medium

For the first three problems, separate the variables by letting $u(x,t) = X(x)T(t)$. The boundary conditions $u(0,t) = u(L,t) = 0$ leads to eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

for the separation constant, and corresponding eigenfunctions

$$X_n(x) = \sin(n\pi x/L).$$

Corresponding solutions for T are

$$T_n(t) = e^{-n^2\pi^2 kt/L^2}.$$

Solutions of these problems therefore all have the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L) e^{-kn^2\pi^2 t/L^2},$$

in which c_n is determined by the initial condition $u(x,0) = f(x)$ by

$$c_n = \frac{2}{L}\int_0^L f(\xi)\sin(n\pi\xi/L)\,d\xi.$$

Therefore, for these problems, all we need do is evaluate these integrals for the coefficients.

1. With $f(x) = x(L-x)$,

$$c_n = \frac{2}{L}\int_0^L \xi(L-\xi)\sin(n\pi\xi/L)\,d\xi = \frac{4L^2}{n^3\pi^3}(1-(-1)^n).$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4L^2}{n^3\pi^3}(1-(-1)^n)\sin(n\imath x/L)e^{-kn^2\pi^2 t/L^2}.$$

We can also observe that

$$1-(-1)^n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd,} \end{cases}$$

so the solution can also be written by summing over just the odd positive integers. This can be done by replacing n with $2n-1$ in the summation:

$$u(x,t) = \frac{8L^2}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}\sin((2n-1)\pi x/L)e^{-k(2n-1)^2\pi^2 t/L^2}.$$

3. Here $k=3$ and $f(x) = L(1-\cos(2\pi x/L))$. Compute

$$c_n = \frac{2}{L}\int_0^L L(1-\cos(2\pi\xi/L))\sin(n\pi\xi/L)\,d\xi = \begin{cases} \frac{8((-1)^n-1)}{n\pi(n^2-4)} & \text{if } n \neq 2, \\ 0 & \text{for } n = 2, \end{cases}$$

Because $(-1)^n - 1 = 0$ if n is even, and -2 if n is odd, we actually have

$$c_n = -\frac{16L}{n\pi(n^2-4)}$$

for $n = 1, 3, 5, \cdots$. We can write the solution as

$$u(x,t) = -\frac{16L}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)((2n-1)^2-4)}\sin((2n-1)\pi x/L)e^{-3(2n-1)^2\pi^2 t/L^2}.$$

Problems 4–7 have insulated boundary conditions, so separation of variables by putting $u(x,t) = X(x)T(t)$ leads to eigenvalues and eigenfunctions $\lambda_0 = 1, X_1 = 1$ and, for $n = 1, 2, \cdots$,

$$\lambda_n = \frac{n^2\pi^2}{L^2}, X_n(x) = \cos(n\pi x/L).$$

We also find that

$$T_n(t) = e^{-kn^2\pi^2 t/L^2}$$

as in the case of boundary conditions $u(0,t) = u(L,t) = 0$. Now the solution has the form

$$u(x,t) = \frac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n\cos(n\pi x/L)e^{-kn^2\pi^2 t/L^2},$$

where

$$c_n = \frac{2}{L}f(\xi)\cos(n\pi\xi/L)\,d\xi.$$

5. Now $k = 4$, $L = 2\pi$ and $f(x) = x(2\pi - x)^2$. The coefficients are

$$c_0 = \frac{1}{\pi}\int_0^{2\pi} \xi(2\pi - \xi)^2\, d\xi = \frac{4}{3}\pi^3$$

and, for $n = 1, 2, \cdots$,

$$\begin{aligned} c_n &= \frac{1}{\pi}\int_0^{2\pi} \xi(2\pi - \xi)^2 \cos(n\xi/2)\, d\xi \\ &= -\frac{16}{\pi n^4}(n^2\pi^2 - 6(1 - (-1)^n)). \end{aligned}$$

The solution is

$$\begin{aligned} u(x,t) &= \frac{2}{3}\pi^3 \\ &\quad - \sum_{n=1}^{\infty} \frac{16}{\pi}\frac{n^2\pi^2 - 6(1-(-1)^n)}{n^4}\cos(nx/2)e^{-n^2 t}. \end{aligned}$$

7. In this problem $L = 6, k = 2$ and $f(x) = x\cos(\pi x/4)$. The coefficients in the solution are

$$c_0 = \frac{1}{3}\int_0^6 \xi\cos(\pi\xi/4)\, d\xi = -\frac{8}{3}\frac{2 + 3\pi}{\pi^2},$$

and, for $n = 1, 2, \cdots$,

$$\begin{aligned} c_n &= \frac{1}{6}\int_0^6 \xi\cos(\pi\xi/4)\cos(n\pi\xi/6)\, d\xi \\ &= \frac{24(-18 - 8n^2 - 27\pi(-1)^n + 12\pi n^2(-1)^n)}{\pi^2(2n-3)^2(2n+3)^2}. \end{aligned}$$

The solution is

$$\begin{aligned} u(x,t) &= -\frac{4}{3}\frac{2+3\pi}{\pi^2} \\ &\quad + \sum_{n=1}^{\infty} c_n\cos(n\pi x/6)e^{-n^2\pi^2 t/18}. \end{aligned}$$

9. The initial-boundary value problem for the temperature function is

$$\begin{aligned} u_t &= u_{xx} \text{ for } 0 < x < L, t > 0, \\ u(0,t) &= u_x(L,t) = 0, \\ u(x,0) &= \frac{Bx}{L}. \end{aligned}$$

Separate variables by putting $u(x,t) = X(x)T(t)$ to obtain

$$X'' + \lambda X = 0; X(0) = X'(L) = 0$$

and

$$T'' + \lambda k T = 0.$$

By taking cases on λ, we find the eigenvalues and corresponding eigenfunctions:

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2 \text{ and } X_n(x) = \sin((2n-1)\pi x/2L).$$

Further,

$$T_n(x) = e^{-k(2n-1)^2\pi^2 t/4L^2}.$$

The solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin((2n-1)\pi x/2L) e^{-k(2n-1)^2\pi^2 t/4L^2}.$$

The coefficients are

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L \frac{B}{L}\xi \sin((2n-1)\pi\xi/2L)\,d\xi \\ &= \frac{-8B}{\pi^2(2n-1)^2}(-1)^n. \end{aligned}$$

The solution is

$$u(x,t) = -\frac{8B}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin((2n-1)\pi x/2L)e^{-k(2n-1)^2\pi^2 t/4L^2}.$$

11. Make the transformation $u(x,t) = e^{\alpha x+\beta t}v(x,t)$. Following the discussion of the text, let $\alpha = -A/2 = -4/2 = -2$ and $\beta = k(B - A2/4) = -2$ also, so

$$u(x,t) = e^{-2x-2t}v(x,t)$$

and v is the solution of the problem

$$\begin{aligned} v_t &= v_{xx} \text{ for } 0 < x < \pi, t > 0, \\ v(0,t) &= v(\pi,t) = 0, \\ v(x,0) &= e^{2x}u(x,0) = xe^{2x}(\pi - x). \end{aligned}$$

This has the solution

$$v(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx)e^{-n^t t},$$

where

$$\begin{aligned} c_n &= \frac{2}{\pi}\int_0^{\pi} \xi e^{2\xi}(\pi-\xi)\sin(n\xi)\,d\xi \\ &= -\frac{4}{\pi(4+n^2)^3}\left[24n - 2n^3 + 16n\pi + 4n^3\pi - 24ne^{2\pi}(-1)^n\right. \\ &\left. +2e^{2\pi}n^3(-1)^n + 16n\pi e^{2\pi}(-1)^n + 4n^3\pi e^{2\pi}(-1)^n\right]. \end{aligned}$$

The solution of the original problem is

$$u(x,t) = e^{-2x-2t}v(x,t).$$

13. Here we have $A = 6, B = 0, k = 1, L = \pi$ and $u(x,0) = f(x) = x(\pi - x)$. Let $\alpha = 3$ and $\beta = -9$ and let

$$u(x,t) = e^{3x-9t}v(x,t).$$

The $v(x,t)$ satisfies

$$\begin{aligned} v_t &= v_{xx} \text{ for } 0 < x < \pi, t > 0, \\ v(0,t) &= v(\pi,t) = 0, \\ v(x,0) &= e^{-3x}f(x) = x(\pi - x)e^{-3x}. \end{aligned}$$

The solution of this problem is

$$v(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx)e^{-n^2 t},$$

where

$$\begin{aligned} c_n &= \frac{2}{\pi}\int_0^{\pi} e^{-3\xi}\xi(\pi - \xi)\sin(n\xi)\,d\xi \\ &= \frac{4n}{\pi(n^2+9)^3}(1 - (-1)^n e^{-3\pi})(3\pi(n^2+9) + n^2 - 27). \end{aligned}$$

The original problem has the solution

$$v(x,t) = e^{3x-9t}v(x,t).$$

15. Let $u(x,t) = v(x,t) + \psi(x)$. To get a standard problem for $v(x,t)$, choose $\psi(x)$ so that $\psi'' = 0$ and

$$\psi(0) = T, \psi(L) = 0.$$

Then

$$\psi(x) = \frac{T}{L}(L - x)$$

The problem for v is

$$\begin{aligned} v_t &= kv_{xx} \text{ for } 0 < x < L, t > 0, \\ v(0,t) &= v(L,t) = 0, \\ v(x,0) &= u(x,0) - \psi(x) = x(L-x)^2 - \frac{T}{L}(L - x). \end{aligned}$$

This has the solution

$$v(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L)e^{-kn^2\pi^2 t/L^2},$$

where

$$c_n = \frac{2}{L}\int_0^L \left[\xi(1-\xi)^2 - \frac{T}{L}(L-\xi)\right]\sin(n\pi x/L)\,d\xi$$
$$= \frac{2}{n^3\pi^3}\left[-n^2\pi^2 T + 4L^3 + 2L^3(-1)^n\right].$$

17. Let $u(x,t) = v(x,t) + h(x)$ and substitute this into the initial-boundary value problem to choose $h(x)$ and obtain a standard problem for $v(x,t)$. We find that

$$h(x) = T\left(1 - \frac{x}{L}\right)$$

and the problem for $v(x,t)$ is

$$v_t = 9v_{xx} \text{ for } 0 < x < L, t > 0,$$
$$v(0,t) = v(L,t) = 0,$$
$$v(x,0) = -T\left(1 - \frac{x}{L}\right).$$

This has the solution

$$v(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L)e^{-9n^2\pi^2 t/L^2},$$

where

$$c_n = \frac{2}{L}\int_0^L -T\left(1 - \frac{\xi}{L}\right)\sin(n\pi\xi/L)\,d\xi = -\frac{2T}{n\pi}.$$

Then

$$u(x,t) = T\left(1 - \frac{x}{L}\right) - \frac{2T}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi x/L)e^{-9n^2\pi^2 t/L^2}.$$

5.2 The Heat Equation With a Forcing Term $F(x,t)$

In Problems 1–5, notation of the text is used for $B_n(t)$, b_n and $T_n(t)$. Note that the second term in the solution for $u(x,t)$ is the solution to the problem without the forcing.

1. Here $k = 4, L = \pi,\ f(x) = x(\pi - x)$ and $F(x,t) = t$. We need

$$B_n(t) = \frac{2}{\pi}\int_0^{\pi} t\sin(n\xi)\,d\xi = \frac{2t}{n\pi}(1-(-1)^n),$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(\xi)\sin(n\xi)\,d\xi = \frac{4}{n\pi^3}(1-(-1)^n),$$

and

$$\begin{aligned} T_n(t) &= \int_0^t e^{-4n^2(t-\tau)} B_n(\tau)\,d\tau + b_n e^{-4n^2 t} \\ &= \frac{1}{8\pi n^5}(1-(-1)^n)(-1+4n^2t+e^{-4n^2t}). \end{aligned}$$

The solution is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \frac{1}{8\pi n^5}(1-(-1)^n)(-1+4n^2t+e^{-4n^2t})\sin(nx) \\ &+ \sum_{n=1}^{\infty} \frac{4}{\pi n^3}(1-(-1)^n)\sin(nx)e^{-4n^2t}. \end{aligned}$$

3. First,

$$\begin{aligned} B_n(t) &= \frac{2}{5}\int_0^5 t\cos(\xi)\sin(n\pi\xi/5)\,d\xi \\ &= \frac{2t}{n^2\pi^2-25}((-1)^{n+1}(n\pi+5)+n\pi), \end{aligned}$$

$$b_n = \frac{2}{5}\int_0^5 \xi^2(5-\xi)\sin(n\pi\xi/5)\,d\xi = \frac{500}{n^3\pi^3}((-1)^{n+1}-1)$$

and

$$T_n(t) = \frac{50(1-cos(5)(-1)^n)}{n^3\pi^3(n^2\pi^2-25)}\left(n^2\pi^2t-25+25e^{-n^2\pi^2t/25}\right).$$

The solution is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \frac{50(1-\cos(5)(-1)^n)}{n^3\pi^3(n^2\pi^2-25)}(n^2\pi^2t-25+25e^{-n^2\pi^2t/25})\sin(n\pi x/5) \\ &+ \sum_{n=1}^{\infty} \frac{500}{n^3\pi^3}((-1)^{n+1}-1)\sin(n\pi x/5)e^{-n^2\pi^2t/25}. \end{aligned}$$

Sometimes a graphic can display features of a solution. This is done for Problem 3 (Section 5.2). Figure 5.1 shows part of a surface plot of the solution without the forcing term, and Figure 5.2 shows the solution with the forcing term. In Figure 5.1, the temperature decreases quickly to zero, without the introduction of new energy, while in Figure 5.2 this does not occur.

5. First compute

$$B_n(t) = \frac{2}{3}\int_0^3 \xi t\sin(n\pi\xi/3)\,d\xi = \frac{6t}{n\pi}(-1)^{n+1},$$

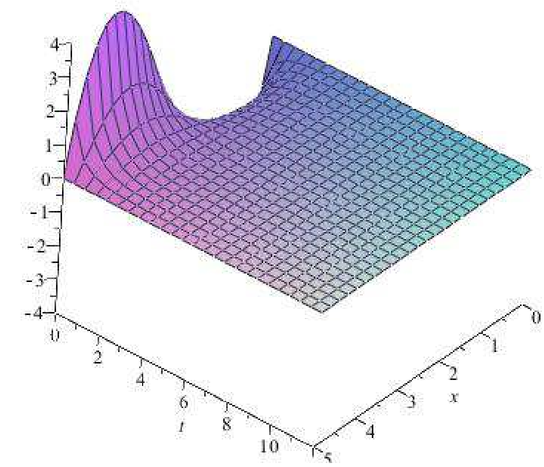

Figure 5.1: Solution surface for Problem 3, without effects of the forcing term included.

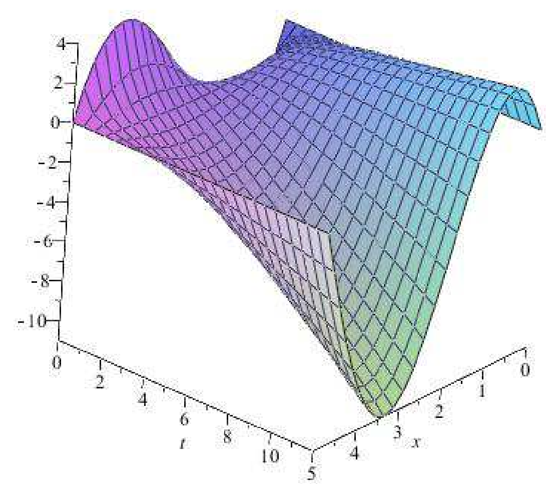

Figure 5.2: Solution surface for Problem 3, including effects of the forcing term.

$$b_n = \frac{2}{3}\int_0^3 K\sin(n\pi\xi/3)\,d\xi = \frac{2K}{n\pi}(1-(-1)^n),$$

and

$$T_n(t) = \frac{27(-1)^{n+1}}{128n^5\pi^5}(16n^2\pi^2 - 9 + 9e^{-16n^2\pi^2 t/9}).$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty}\frac{27(-1)^{n+1}}{128n^5\pi^5}(16n^2\pi^2 - 9 + 9e^{-16n^2\pi^2 t/9})\sin(n\pi x/3)$$
$$+\sum_{n=1}^{\infty}\frac{2K}{n\pi}(1-(-1)^n)\sin(n\pi x/5)e^{-16n^2\pi^2 t/9}.$$

5.3 The Heat Equation on the Real Line

1. With $f(x) = e^{-|x|}$, compute

$$a_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-|\xi|}\cos(\omega\xi)\,d\xi = \frac{8}{\pi}\frac{1}{16+\omega^2}$$

and $b_\omega = 0$ because $f(x)$ is an even function on the real line. The solution is

$$u(x,t) = \frac{8}{\pi}\int_0^{\infty}\frac{1}{16+\omega^2}\cos(\omega x)e^{-\omega^2 kt}.$$

The solution can also be written in the form

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty} e^{-|\xi|}e^{-(x-\xi)^2/4kt}\,d\xi.$$

3. The coefficients are

$$a_\omega = \frac{1}{\pi}\int_0^4 \xi\cos(\omega\xi)\,d\xi = \frac{1}{\pi\omega^2}(4\omega\sin(4\omega) + \cos(4\omega) - 1)$$

and

$$b_\omega = \frac{1}{\pi}\int_0^4 \xi\sin(\omega\xi)\,d\xi = \frac{1}{\pi\omega^2}(\sin(4\omega) - 4\omega\cos(4\omega)).$$

The solution is

$$u(x,t) = \int_0^{\infty}(a_\omega\cos(\omega x) + b_\omega\sin(\omega x))e^{-\omega^2 kt}\,d\omega.$$

We can also write

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}}\int_0^4 \xi e^{-(x-\xi)^2/4kt}\,d\xi.$$

In each of Problems 5–8, the solution has the form

$$u(x,t) = \int_0^\infty [a_\omega \cos(\omega x) + b_\omega \sin(\omega x)] e^{-\omega^2 kt}$$

and just a_ω and b_ω are given

5. $a_\omega = 0$ because $f(x)$ is an even function, and

$$b_\omega = \frac{4(1-\cos(\omega))}{\pi\omega}.$$

7. Each $b_\omega = 0$, while

$$a_\omega = \frac{2\cos(\pi\omega/2)}{\pi(1-\omega^2)}.$$

9. Let

$$F(x) = \int_0^\infty e^{-\zeta^2}\cos(x\zeta)\,d\zeta.$$

By differentiating under the integral sign and then integrating by parts, we obtain

$$\begin{aligned} F'(x) &= \int_0^\infty -\zeta e^{-\zeta^2}\sin(x\zeta)\,d\zeta \\ &= \frac{1}{2}\zeta e^{-\zeta^2}\sin(x\zeta)\Big]_0^\infty - \frac{1}{2}x\int_0^\infty e^{-\zeta^2}\cos(x\zeta)\,d\zeta \\ &= -\frac{1}{2}x\int_0^\infty e^{-\zeta^2}\cos(x\zeta)\,d\zeta \\ &= -\frac{1}{2}xF(x). \end{aligned}$$

The linear differential equation

$$F'(x) + \frac{1}{2}xF(x) = 0$$

has the general solution

$$F(x) = ke^{-x^2/4},$$

with k an arbitrary constant. However, we also know that

$$F(0) = \int_0^\infty e^{-\zeta^2}\,d\zeta = \frac{1}{2}\sqrt{\pi},$$

an integral that can be found in tables and is widely used in statistics. Therefore

$$F(x) = \int_0^\infty e^{-\zeta^2}\cos(x\zeta)\,d\zeta = \frac{1}{2}\sqrt{\pi}e^{-x^2/4}.$$

Now let $x = \alpha/\beta$ to obtain

$$\int_0^\infty e^{-\zeta^2} \cos\left(\frac{\alpha\zeta}{\beta}\right) d\zeta = \frac{1}{2}\sqrt{\pi} e^{-\alpha^2/4\beta^2}.$$

Finally, this integral is half the value of the integral of the same function from $-\infty$ to ∞, so

$$\int_{-\infty}^\infty e^{-\zeta^2} \cos\left(\frac{\alpha\zeta}{\beta}\right) d\zeta = \sqrt{\pi} e^{-\alpha^2/4\beta^2}.$$

This is equation (5.17).

5.4 The Heat Equation on a Half-Line

In each of Problems 1–4, the initial condition is $u(x, 0) = f(x)$ and the solution has the form

$$u(x, t) = \int_0^\infty b_\omega \sin(\omega x) e^{-k\omega^2 t}\, d\omega,$$

where

$$b_\omega = \frac{2}{\pi} \int_0^\infty f(\xi) \sin(\omega\xi)\, d\xi.$$

1. Compute

$$b_\omega = \frac{2}{\pi} \int_0^\infty e^{-\alpha\xi} \sin(\omega\xi)\, d\xi = \frac{2}{\pi} \frac{\omega}{\omega^2 + \alpha^2},$$

so the solution is

$$u(x, t) = \frac{2}{\pi} \int_0^\infty \frac{\omega}{\omega^2 + \alpha^2} \sin(\omega x) e^{-k\omega^2 t}\, d\omega.$$

3. The coefficients are

$$b_\omega = \frac{2}{\pi} \int_0^h \sin(\omega\xi)\, d\xi = \frac{2}{\pi} \frac{1 - \cos(h\omega)}{\omega},$$

and the solution is

$$u(x, t) = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos(h\omega)}{\omega} \sin(\omega x) e^{-k\omega^2 t}\, d\omega.$$

In Problems 5–8, the heat equation is to be solved on the half-line $x > 0$, but the initial condition is now the insulation condition $u_x(x, 0) = f(x)$. Now the solution is

$$u(x, t) = \int_0^\infty a_\omega \cos(\omega x) e^{-\omega^2 k t}\, d\omega,$$

where

$$a_\omega = \frac{2}{\pi} \int_0^\infty f(\xi) \cos(\omega\xi)\, d\xi.$$

We will just give a_ω for each problem.

5.

$$\begin{aligned} a_\omega &= \frac{2}{\pi}\int_0^4 \xi(\xi+1)\cos(\omega\xi)\,d\xi \\ &= \frac{2}{\pi\omega^3}(20\omega^2\sin(4\omega) - \omega - 2\sin(4\omega) + 9\omega\cos(4\omega)) \end{aligned}$$

7.

$$\begin{aligned} a_\omega &= \frac{2}{\pi}\int_5^9 4\cos(\omega\xi)\,d\xi \\ &= \frac{8(\sin(9\omega) - \sin(5\omega))}{\pi\omega} \end{aligned}$$

5.5 The Two-Dimensional Heat Equation

With the condition of zero initial temperature on the sides of the rectangle, and initial temperature $u(x, y, 0) = f(x, y)$, the solution is

$$\sum_{n=1}^{\infty}\sum_{n=1}^{\infty} c_{nm}\sin(n\pi x/L)\sin(m\pi y/K)e^{-\alpha_{nm}kt},$$

where

$$\alpha_{nm} = \frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{K^2}$$

and

$$c_{nm} = \frac{4}{LK}\int_0^L\int_0^K f(\xi,\eta)\sin(n\pi\xi/L)\sin(m\pi\eta/K)\,d\eta\,d\xi.$$

In the problems we will give the values of α_{nm} and c_{nm} for the particular initial temperature function.

1. Here $k = 1$, L and K are positive numbers, and

$$f(x, y) = x(L - x)y^2(K - y).$$

Because $f(x, y)$ is a product of a function of x and a function of y, we have

$$\begin{aligned} c_{nm} &= \frac{4}{LK}\left(\int_0^L \xi(L-\xi)\sin(n\pi\xi/L)\,d\xi\right)\left(\int_0^K \eta^2(K-\eta)\sin(m\pi\eta/K)\,d\eta\right) \\ &= -\frac{16}{L^2K^3}n^3m^3\pi^6(1-(-1)^n)(1+2(-1)^m)) \end{aligned}$$

and

$$\alpha_{nm} = \frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{K^2}.$$

3. Now $k = 1$ and $L = K = \pi$, and

$$c_{nm} = \frac{4}{\pi^2}\left(\int_0^{\pi} \sin(\xi)\sin(n\xi)\,d\xi\right)\left(\int_0^{\pi} \eta\cos(\eta/2)\cos(m\eta)\,d\eta\right).$$

Now,

$$\int_0^{\pi} \sin(\xi)\sin(n\xi)\,d\xi = \begin{cases} \pi/2 & \text{if } n = 1, \\ 0 & \text{for } n = 2, 3, \cdots. \end{cases}$$

Therefore, in the double summation for $u(x, y, t)$, we have only c_{1m} terms and the summation is for $m = 1$ to ∞. Completing the computation of the integrations with respect to η, we obtain

$$c_{1,m} = \frac{32m(-1)^{m+1}}{(4m^2 - 1)^2}.$$

Further,

$$\alpha_{1m} = 1 + m^2.$$

The solution is

$$u(x, y, t) = \sum_{m=1}^{\infty} \frac{32m(-1)^{m+1}}{(4m^2 - 1)^2} \sin(x)\sin(my)e^{-(1+m^2)t}.$$

Chapter 6

The Wave Equation

6.1 Wave Motion on an Interval

For each of Problems 1–8, the problem involves the wave equation on $[0, L]$, with fixed ends, initial position $y(x, 0) = f(x)$, and initial velocity $y_t(x, 0) = g(x)$. The solution is

$$y(x,t) = \sum_{n=1}^{\infty} [a_n \cos(n\pi ct/L) + b_n \sin(n\pi ct/L)] \sin(n\pi x/L),$$

where

$$a_n = \frac{2}{L}\int_0^L f(\xi)\sin(n\pi\xi/L)\,d\xi$$

and

$$b_n = \frac{2}{n\pi c}\int_0^L g(\xi)\sin(n\pi\xi/L)\,d\xi.$$

1. Here $c = 1, L = 2$, the initial position is given by $f(x) = 0$, and the initial velocity is

$$g(x) = \begin{cases} 2x & \text{for } 0 \le x \le 1, \\ 0 & \text{for } 1 < x \le 2. \end{cases}$$

In the general expression for the solution, then, we have $a_n = 0$ for $n = 1, 2, \cdots$ and

$$\begin{aligned} b_n &= \frac{2}{n\pi}\int_0^2 g(\xi)\sin(n\pi\xi/2) \\ &= \frac{2}{n\pi}\int_0^1 2\xi\sin(n\pi\xi/2)\,d\xi \\ &= \frac{8}{n^3\pi^3}[2\sin(n\pi/2) - n\pi\cos(n\pi/2)]. \end{aligned}$$

The solution is

$$y(x,t) = \frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}[2\sin(n\pi/2) - n\pi\cos(n\pi/2)]\sin(n\pi x/2)\sin(n\pi t/2).$$

3. Each $a_n = 0$ and

$$b_b = \frac{1}{n\pi}\int_0^3 \xi(3-\xi)\sin(n\pi x/3)\,d\xi = \frac{54}{n^4\pi^4}(1-(-1)^n).$$

The solution is

$$y(x,t) = \sum_{n=1}^{\infty}\frac{54}{n^4\pi^4}(1-(-1)^n)\sin(n\pi x/3)\sin(2n\pi t/3).$$

Because $(1-(-1)^n)$ is 2 if n is odd, and zero if n is even, we can also write the solution by summing only over the odd positive integers. This is achieved by replacing n with $2n-1$ in the expression being summed, and replacing each $(1-(-1)^n)$ with 2:

$$y(x,t) = \sum_{n=1}^{\infty}\frac{108}{(2n-1)^4\pi^4}\sin((2n-1)\pi x/3)\sin(2(2n-1)\pi t/3).$$

5. The solution is

$$y(x,t) = \frac{24}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin((2n-1)x/2)\cos((2n-1)\sqrt{2}t).$$

7. The solution is

$$y(x,t) = -\frac{32}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin((2n-1)\pi x/2)\cos((3(2n-1)\pi t/2)$$
$$+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}[\cos(n\pi/4) - \cos(n\pi/2)]\sin(n\pi x/2)\sin(3n\pi t/2).$$

9. Let $y(x,t) = Y(x,t) + \psi(x)$ and substitute into the wave equation

$$y_{tt} = Y_{tt} = 3y_{xx} + 2x = 3Y_{xx} + 3\psi''(x) + 2x.$$

Choose $\psi(x)$ so that $3\psi''(x) + 2x = 0$. This means that

$$\psi(x) = -\frac{1}{9}x^3 + cx + d.$$

Now,

$$y(0,t) = Y(0,t) + \psi(0) = 0$$

so let $d = 0$ to have $\psi(0) = 0$. Then $Y(0,t) = 0$.

Next

$$y(2,t) = Y(2,t) + \psi(2) = Y(2,t) - \frac{8}{9} + 2c = 0.$$

We will have $Y(2,t) = 0$ if $c = 4/9$. This means that

$$\psi(x) = -\frac{1}{9}x^3 + \frac{4}{9}x = \frac{1}{9}x(4 - x^2).$$

The problem for $Y(x,t)$ is

$$\begin{aligned} Y_{tt} &= 3Y_{xx} \text{ for } 0 < x < 2, t > 0, \\ Y(0,t) &= Y(2,t) = 0, \\ Y(x,0) &= y(x,0) - \psi(x) = \frac{1}{9}x(x^2 - 4). \end{aligned}$$

The solution for $Y(x,t)$ is

$$Y(x,t) = \sum_{n=1}^{\infty} \frac{32}{3} \frac{(-1)^n}{n^3\pi^3} \sin(n\pi x/2)\cos(n\pi\sqrt{t}/2).$$

The original problem has the solution

$$y(x,t) = Y(x,t) + \frac{1}{9}x(4 - x^2).$$

11. Let $y(x,t) = Y(x,t) + \psi(x)$. Substitute this into the wave equation to get

$$y_{tt} = Y_{tt} = y_{xx} = Y_{xx} + \psi''(x) - \cos(x).$$

This will give us $Y_{tt} = Y_{xx}$ if $\psi''(x) = \cos(x)$, which means that

$$\psi(x) = -\cos(x) + cx + d.$$

Now

$$y(0,t) = 0 = Y(0,t) + \psi(0) = -1 + d.$$

This will give us $Y(0,t) = 0$ if $d = 1$. Next,

$$y(2\pi,t) = 0 = Y(2\pi,t) - \cos(2\pi) + 2\pi c + 1.$$

This will give us $Y(2\pi,0) = 0$ if $c = 0$. Then

$$\psi(x) = -\cos(x) + 1.$$

Finally,

$$y(x,0) = Y(x,0) - \cos(x) + 1 = 0$$

implies that $Y(x,0) = \cos(x) - 1$. And

$$y_t(x,0) = Y_t(x,0) = x.$$

The problem for $Y(x,t)$ is:

$$\begin{aligned} Y_{tt} &= Y_{xx} \text{ for } 0 < x < 2\pi, t > 0, \\ Y(0,t) &= Y(2\pi,t) = 0, \\ Y(x,0) &= \cos(x) - 1, Y_t(x,0) = x. \end{aligned}$$

This has a solution of the form

$$Y(x,t) = \sum_{n=1}^{\infty} [a_n \cos(nt/2) + b_n \sin(nt/2)] \sin(nx/2),$$

where

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} (\cos(\xi) - 1) \sin(n\xi/2)\, d\xi \\ &= \begin{cases} \frac{16}{n\pi(n^2-4)} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases} \end{aligned}$$

and

$$b_n = \frac{2}{\pi} \int_0^{2\pi} \xi \sin(n\xi/2)\, d\xi = \frac{8}{n^2}(-1)^{n+1}.$$

These coefficients determine $Y(x,t)$, and then $y(x,t) = Y(x,t) + 1 - \cos(x)$.

13. Let $y(x,t) = Y(x,t) + \psi(x)$ and substitute this into the wave equation of the problem to choose $\psi(x)$ so that

$$7\psi''(x) + e^{-x} = 0, \psi(0) = \psi(2) = 0.$$

This leads to

$$\psi(x) = -\frac{1}{7}e^{-x} + \frac{1}{14}(e^{-2} - 1)x + \frac{1}{7}.$$

Now

$$\begin{aligned} Y_{tt} &= 7Y_{xx} \text{ for } 0 < x < 2, t > 0, \\ Y(0,t) &= Y(2,t) = 0, \\ Y(x,0) &= -\psi(x), Y_t(x,0) = 5x. \end{aligned}$$

This has the solution

$$Y(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos(n\pi\sqrt{7}t/2) + b_n \sin(n\pi\sqrt{7}t/2)\right] \sin(n\pi x/2),$$

where

$$\begin{aligned} a_n &= \int_0^2 \left(\frac{1}{7}e^{-\xi} - \frac{1}{14}(e^{-2} - 1)\xi - \frac{1}{7}\right) \sin(n\pi\xi/2)\, d\xi \\ &= \frac{2}{7n\pi(4 + n^2\pi^2)}(-4 - n^2\pi^2 e^{-2}(-1)^n + e^{-1}n^2\pi^2(-1)^n + 4e^{-1}(-1)^n), \end{aligned}$$

and

$$b_n = \frac{2}{n\pi\sqrt{7}} \int_0^2 5\xi \sin(n\pi\xi/2)\, d\xi = \frac{40(-1)^{n+1}}{n^2\pi^2\sqrt{7}}.$$

These coefficients determine $Y(x,t)$, and then $y(x,t) = Y(x,t) + \psi(x)$.

15. (a) Substitute $y(x,t) = X(x)T(t)$ into the fourth-order differential equation to get

$$X^{(4)} - \lambda X = 0, T'' + \lambda a^4 \lambda T = 0,$$

with λ the separation constant. Note - by rearranging terms differently, we can reach different separated equations for X and T. For example, we could have kept the a^4 factor with the X terms.

(b) Consider cases on λ, noting that the boundary conditions are

$$X''(0) = X''(\pi) = X^{(3)}(0) = X^{(3)}(\pi) = 0.$$

Case 1 - Suppose $\lambda = 0$. Then $X^{(4)}(x) = 0$ and four integrations give us

$$X(x) = A + Bx + Cx^2 + DX^3.$$

The boundary conditions force $C = D = 0$, while A and B are arbitrary. Therefore 0 is an eigenvalue of this problem, with eigenfunctions $X_0(x) = A + Bx$, with A and B not both zero. In this case solutions for T are $T(t) = \alpha + \beta t$.

Case 2 - Suppose $\lambda < 0$. The notation is simplified if we set $\lambda = -4\alpha^4$, with $\alpha > 0$. The differential equation for X is

$$X^{(4)} + 4\alpha^4 X = 0,$$

with characteristic equation $r^4 + 4\alpha^4 = 0$. This has roots

$$(1+i)\alpha, (1-i)\alpha, (-1+i)\alpha \text{ and } (-1-i)\alpha.$$

In this case solutions are

$$X(x) = e^{\alpha x}(A\cos(\alpha x) + B\sin(\alpha x)) + e^{-\alpha x}(C\cos(\alpha x) + D\sin(\alpha x)).$$

Apply the boundary conditions to this general solution to obtain:

$$\begin{aligned}
&B - D = 0,\\
&A - B - C - D = 0,\\
&-Ae^{\alpha\pi}\sin(\alpha\pi) + Be^{\alpha\pi}\cos(\alpha\pi) + Ce^{-\alpha\pi}\sin(\alpha\pi) - De^{-\alpha\pi}\cos(\alpha\pi) \quad = 0,\\
&-Ae^{\alpha\pi}(\cos(\alpha\pi) + \sin(\alpha\pi)) + Be^{\alpha\pi}(\cos(\alpha\pi) - \sin(\alpha\pi))\\
&-Ce^{-\alpha\pi}(\cos(\alpha\pi) - \sin(\alpha\pi)) - De^{-\alpha\pi}(\cos(\alpha\pi) + \sin(\alpha\pi)) = 0.
\end{aligned}$$

This is a 4×4 homogeneous system of linear algebraic equations. This system has a nontrivial solution if and only if the determinant of the coefficients is zero:

$$\cosh(2\alpha\pi) - \cos(2\alpha\pi) = 0.$$

But this equation is satisfied only by $\alpha = 0$, and in this case $\alpha > 0$. Therefore the system has only the trivial solution $A = B = C = D = 0$, and this problem has no negative eigenvalue.

Case 3 - Suppose $\lambda > 0$, say $\lambda = \alpha^4$ with $\alpha > 0$. Now $X^{(4)} - \alpha^4 X = 0$, and the characteristic equation has roots

$$\alpha, -\alpha, \alpha i, -\alpha i.$$

The general solution is

$$X(x) = A\cos(\alpha x) + B\sin(\alpha x) + C\cosh(\alpha x) + D\sinh(\alpha x).$$

The boundary conditions give us four equations:

$$\begin{aligned}
&-A + C = 0,\\
&-A\cos(\alpha\pi) - B\sin(\alpha\pi) + C\cosh(\alpha\pi) + D\sinh(\alpha\pi) = 0,\\
&-B + D = 0,\\
&A\sin(\alpha\pi) + B\cos(\alpha\pi) + C\sinh(\alpha\pi + D\cosh(\alpha\pi) = 0.
\end{aligned}$$

From the first and third equations, $A = C$ and $B = D$. This reduces the system to the second and fourth equations in two unknowns:

$$\begin{aligned}
C(\cosh(\alpha\pi) - \cos(\alpha\pi)) + D(\cosh(\alpha\pi) - \sin(\alpha\pi)) = 0,\\
C(\sinh(\alpha\pi) + \sin(\alpha\pi)) + D(\cosh(\alpha\pi) - \cos(\alpha\pi)) = 0.
\end{aligned}$$

This 2×2 system has a nontrivial solution if and only if the determinant of the system is nonzero. This requires that

$$\cos(\alpha\pi)\cosh(\alpha\pi) = 1.$$

It may not be obvious, but this equation has infinitely many positive solutions for α (two are 2.499752670 and 0.000000207171091). If these solutions for α are listed $\alpha_1 < \alpha_2 < \cdots$, then $\lambda_n = \alpha_n^4$ is an eigenvalue of the problem. Eigenfunctions then have the form

$$X_n(x) = A\cos(\alpha_n x) + B\sin(\alpha_n x) + C\cosh(\alpha_n x) + D\sinh(\alpha_n x).$$

6.2 Wave Motion in an Unbounded Medium

For the wave equation on a the real line,

$$\begin{aligned}
y_{tt} = c^2 y_{xx} \text{ for } -\infty < x < \infty, t > 0,\\
y(x,0) = f(x), y_t(x,0) = 0,
\end{aligned}$$

specifying an initial position but zero initial velocity, a solution can be found very much like the problem for an interval $[0, L]$, with Fourier integrals replacing the Fourier series seen in the bounded interval case. The solution is

$$y(x,t) = \int_0^\infty [a_\omega \cos(\omega x) + b_\omega \sin(\omega x)] \cos(\omega ct)\, d\omega,$$

where

$$a_\omega = \frac{1}{\pi} \int_{-\infty}^\infty f(\xi) \cos(\omega \xi)\, d\xi$$

and

$$b_\omega = \frac{1}{\pi} \int_{-\infty}^\infty f(\xi) \sin(\omega \xi)\, d\xi.$$

If $y(x,0) = 0$ and $y_t(x,0) = g(x)$ (string released without initial displacement, but with initial velocity $g(x)$), then the solution is

$$y(x,t) = \int_0^\infty [\alpha_\omega \cos(\omega x) + \beta_\omega \sin(\omega x)] \sin(\omega ct)\, d\omega,$$

where

$$\alpha_\omega = \frac{1}{\pi \omega c} \int_{-\infty}^\infty g(\xi) \cos(\omega \xi)\, d\xi$$

and

$$\beta_\omega = \frac{1}{\pi \omega c} \int_{-\infty}^\infty g(\xi) \sin(\omega \xi)\, d\xi.$$

If the problem has $f(x)$ and $g(x)$ both nonzero, then the solution is the sum of the solution with zero initial velocity, and the solution with no initial displacement.

1. With $c = 5$, $f(x) = e^{-5|x|}$ and $g(x) = 0$, compute

$$a_\omega = \frac{1}{\pi} \int_{-\infty}^\infty e^{-5|\xi|} \cos(\omega \xi)\, d\xi = \frac{10}{(25+\omega^2)\pi}$$

and $b_\omega = 0$ because $f(x)$ is an even function. The solution is

$$y(x,t) = \frac{10}{\pi} \int_0^\infty \left(\frac{1}{25+\omega^2} \right) \cos(\omega x) \cos(12\omega t)\, d\omega.$$

3. Compute the coefficients to determine the solution

$$y(x,t) = \int_0^\infty \left(-\frac{\sin(\pi\omega)}{2\pi\omega(\omega^2-1)} \right) \sin(\omega x) \sin(4\omega t)\, d\omega.$$

5. The solution is

$$y(x,t) = \int_0^\infty [\alpha_\omega \cos(\omega x) + \beta_\omega \sin(\omega x)] \sin(3\omega t)\, d\omega,$$

where

$$\alpha_\omega = \frac{1}{3\pi\omega}\int_1^\infty e^{-2\xi}\cos(\omega\xi)\,d\xi = \frac{1}{3e^2\pi\omega}\frac{2\cos(\omega)-\omega\sin(\omega)}{4+\omega^2}$$

and

$$\beta_\omega = \frac{1}{3\pi\omega}\int_1^\infty e^{-2\xi}\sin(\omega\xi)\,d\xi = \frac{1}{3e^2\pi\omega}\frac{2\sin(\omega)+\omega\cos(\omega)}{4+\omega^2}.$$

7. The solution for the problem with the given displacement and zero initial velocity is

$$y_1(x,t) = \int_0^\infty [a_\omega\cos(\omega x) + b_\omega\sin(\omega x)]\cos(7\omega t)\,d\omega,$$

where

$$\begin{aligned} a_\omega &= \frac{1}{\pi}\int_{-1}^5 f(\xi)\cos(\omega\xi)\,d\xi \\ &= \frac{\sin(\omega)-2\sin(2\omega)+3\sin(5\omega)}{\pi\omega} \end{aligned}$$

and

$$\begin{aligned} b_\omega &= \frac{1}{\pi}\int_{-1}^5 f(\xi)\sin(\omega\xi)\,d\xi \\ &= \frac{\cos(\omega)+2\cos(2\omega)-3\cos(5\omega)}{\pi\omega}. \end{aligned}$$

The solution for the problem with the given velocity, but zero initial displacement, is

$$y_2(x,t) = \int_0^\infty [\alpha_\omega\cos(\omega x) + \beta_\omega\sin(\omega x)]\sin(7\omega t)\,d\omega,$$

where

$$\begin{aligned} \alpha_\omega &= \frac{1}{7\pi\omega}\int_{-1}^1 e^{-|\xi|}\cos(\omega\xi)\,d\xi \\ &= -\frac{2}{7\pi(1+\omega^2)}(e^{-1}\cos(\omega) - \omega e^{-1}\sin(\omega) - 1) \end{aligned}$$

and

$$\beta_\omega = \frac{1}{7\pi\omega}\int_{-1}^1 e^{-|\xi|}\sin(\omega\xi)\,d\xi = 0.$$

The solution of the problem with initial displacement $f(x)$ and initial velocity $g(x)$ is

$$y(x,t) = y_1(x,t) + y_2(x,t).$$

9. Following the notation of Problems 7 and 8, form $y_1(x,t)$ with coefficients

$$a_\omega = \frac{1}{\pi}\int_{-2}^{2} \xi\cos(\omega\xi)\,d\xi = 0$$

and

$$\begin{aligned} b_\omega &= \frac{1}{\pi}\int_{-2}^{2} \xi\sin(\omega\xi)\,d\xi \\ &= \frac{2\sin(2\omega) - 4\cos(2\omega)}{\pi\omega^2}. \end{aligned}$$

And $y_2(x,t)$ has coefficients

$$\begin{aligned} \alpha_\omega &= \frac{4}{\pi\omega}\int_{-3}^{3} \xi^2\cos(\omega\xi)\,d\xi \\ &= \frac{8}{\pi\omega^4}(9\omega^2\sin(3\omega) - 2\sin(3\omega) + 6\omega\cos(3\omega)) \end{aligned}$$

and $\beta_\omega = 0$.

The solution is

$$\begin{aligned} y(x,t) = &\int_0^\infty a_n\cos(\omega x)\cos(\omega t/4) \\ &+ \int_0^\infty \beta_\omega\sin(\omega x)\sin(\omega t/4). \end{aligned}$$

For the problem on a half-line $[0,\infty)$, there is a boundary condition which we will take to be

$$y(0,t) = 0$$

along with initial conditions

$$y(x,0) = f(x), y_t(x,0) = g(x)$$

for $x > 0$. The solution has the form

$$y(x,t) = \int_0^\infty [A_\omega\cos(\omega ct) + B_\omega\sin(\omega ct)]\sin(n\omega x)\,d\omega,$$

where

$$A_\omega = \frac{2}{\pi}\int_0^\infty f(\xi)\sin(\omega\xi)\,d\xi$$

and

$$B_\omega = \frac{2}{\pi}\int_0^\infty g(\xi)\sin(\omega\xi)\,d\xi.$$

11. Here $A_\omega = 0$ and

$$\begin{aligned} B_\omega &= \frac{2}{3\pi\omega}\int_4^{11} 2\sin(\omega\xi)\,d\xi \\ &= \frac{4(\cos(4\omega)-\cos(11\omega))}{3\pi\omega^2}. \end{aligned}$$

The solution is

$$y(x,t) = \frac{4}{3\pi}\int_0^\infty \frac{\cos(4\omega)-\cos(11\omega)}{\omega^2}\sin(\omega x)\sin(3\omega t)\,d\omega.$$

13. With $g(x) = 0$, $B_\omega = 0$ and

$$\begin{aligned} A_\omega &= \frac{2}{\pi}\int_0^\infty -2e^{-\xi}\sin(\omega\xi)\,d\xi \\ &= -\frac{4\omega}{\pi(1+\omega^2)}. \end{aligned}$$

The solution is

$$y(x,t) = -\frac{4}{\pi}\int_0^\infty \frac{\omega}{1+\omega^2}\sin(\omega x)\cos(6\omega t)\,d\omega.$$

15. Compute

$$\begin{aligned} A_\omega &= \frac{2}{\pi}\int_0^1 f(\xi)\sin(\omega\xi)\,d\xi \\ &= \frac{2\sin(\omega)}{\pi^2-\omega^2} \end{aligned}$$

and

$$\begin{aligned} B_\omega &= \frac{2}{\sqrt{13}\pi\omega}\int_0^4 g(\xi)\sin(\omega\xi)\,d\xi \\ &= \frac{2}{\sqrt{13}\pi\omega^2}(1-2\cos(\omega)+\cos(4\omega)). \end{aligned}$$

The solution is

$$y(x,t) = \int_0^\infty [A_\omega\cos(\sqrt{13}\omega t) + B_\omega\sin(\sqrt{13}\omega t)]\sin(\omega x)\,d\omega.$$

6.3 d'Alembert's Solution and Characteristics

1. The characteristics are the lines $x - t = k_1$ and $x + t = k_2$, with k_1 and k_2 arbitrary real numbers. d'Alembert's solution of the problem is

$$\begin{aligned} y(x,t) &= \frac{1}{2}[(x-t)^2 + (x+t)^2] + \frac{1}{2}\int_{x-t}^{x+t} -\xi \, d\xi \\ &= \frac{1}{2}[x^2 - 2xt + t^2 + x^2 + 2xt + t^2 - \left[\frac{1}{2}\xi^2\right]_{x-t}^{x+t} \\ &= x^2 - xt + t^2. \end{aligned}$$

3. characteristics: $x - 7t = k_1, x + 7t = k_2$;

$$y(x,t) = \frac{1}{2}[\cos(\pi(x-7t)) + \cos(\pi(x+7t))] + t - x^2 t - \frac{49}{3}t^3$$

This solution can also be written

$$y(x,t) = \frac{1}{2}\cos(\pi x)\cos(7\pi t) + t - x^2 t - \frac{49}{3}t^3.$$

5. characteristics: $x - 14t = k_1, x + 14t = k_2$;

$$\begin{aligned} y(x,t) &= \frac{1}{2}\left[e^{x-14t} + e^{x+14t}\right] + xt \\ &= e^x \cosh(14t) + xt. \end{aligned}$$

7. characteristics $x - \sqrt{3}t = k_1, x + \sqrt{3}t = k_2$;

$$\begin{aligned} y(x,t) &= \frac{1}{2}\left[e^{-3|x-\sqrt{3}t|} + e^{-3|x+\sqrt{3}t|}\right] \\ &\quad + \frac{1}{\sqrt{3}}\left[\sin\left(\frac{x+\sqrt{3}t}{2}\right) - \sin\left(\frac{x-\sqrt{3}t}{2}\right)\right] \end{aligned}$$

9. The solution with $y(x,0) = f(x) = \sin(x)$ is

$$y(x,t) = \frac{1}{x}(\sin(x-t) + \sin(x+t)).$$

With $y(x,0) = \sin(x) + \epsilon$, the solution is

$$y_\epsilon(x,t) = \frac{1}{2}\sin(x-t) + \epsilon + \sin(x+t) + \epsilon = y(x,t) + \epsilon.$$

11. Now

$$\begin{aligned} y(x,t) &= e^{-3(x-ct)} + \sin(4(x+ct) \\ &= e^{-3x}e^{3ct} + \sin(4(x+ct)). \end{aligned}$$

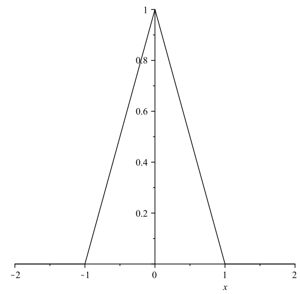

Figure 6.1: $y(x,0)$, Problem 13.

Then

$$\begin{aligned} y_x &= -3e^{-3x}e^{3ct} + 4\cos(4(x+ct)), \\ y_{xx} &= 9e^{-3x}e^{3ct} - 16\sin(4(c+ct)), \\ y_t &= 3ce^{-3x}e^{3ct} + 4c\cos(4(x+ct)), \\ y_{tt} &= 9c^2e^{-3x}e^{3ct} - 16c^2\sin(4(x+ct)). \end{aligned}$$

It is easy to check that $y_{tt} = c^2 y_{xx}$.

In each of Problems 12–17, with $c = 1$ and $g(x) = 0$, the forward wave is $F(x,t) = \frac{1}{2}f(x-t)$ and the backward wave is $B(x,t) = \frac{1}{2}f(x+t)$. The solution is

$$y(x,t) = F(x,t) + B(x,t).$$

13. Figures 6.1–6.6 show graphs of $y(x,t)$ for times

$$t = 0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, 1, \frac{3}{2}.$$

15. Figures 6.7–6.12 show graphs of $y(x,t)$ for

$$t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{3}{2}.$$

17. Figure 6.13 shows a graph of $y(x,0)$, while Figures 6.14 - 6.19 show $y(x,t)$ for times

$$t = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{5}{2}.$$

19. We know that

$$y(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi$$

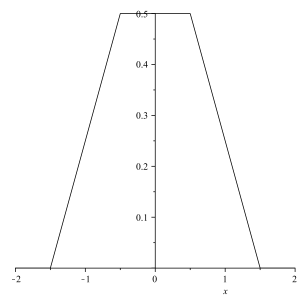

Figure 6.2: $y(x, 1/2)$, Problem 13.

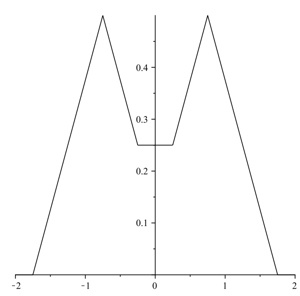

Figure 6.3: $y(x, 3/4)$, Problem 13.

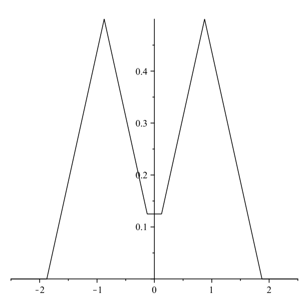

Figure 6.4: $y(x, 7/8)$, Problem 13.

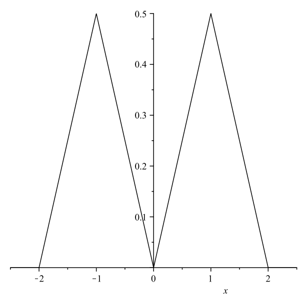

Figure 6.5: $y(x, 1)$, Problem 13.

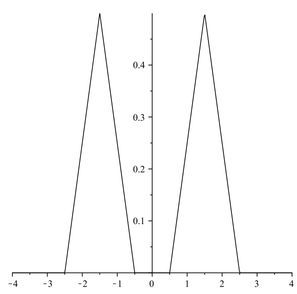

Figure 6.6: $y(x, 3/2)$, Problem 13.

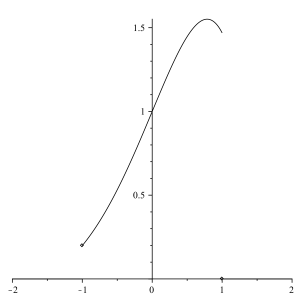

Figure 6.7: $y(x, 0)$, Problem 15.

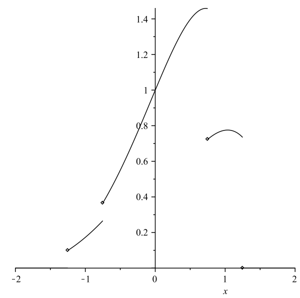

Figure 6.8: $y(x, 1/4)$, Problem 15.

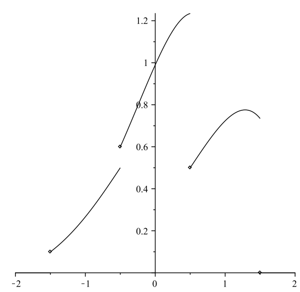

Figure 6.9: $y(x, 1/2)$, Problem 15.

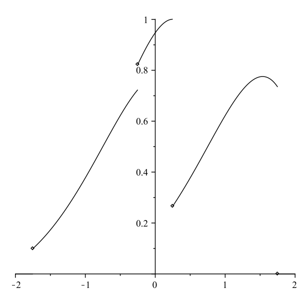

Figure 6.10: $y(x, 3/4)$, Problem 15.

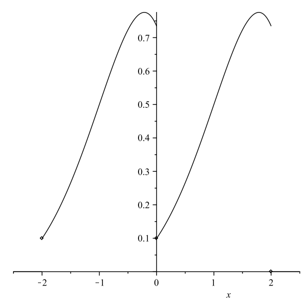

Figure 6.11: $y(x, 1)$, Problem 15.

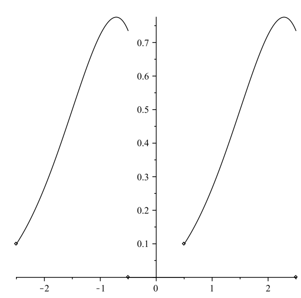

Figure 6.12: $y(x, 3/2)$, Problem 15.

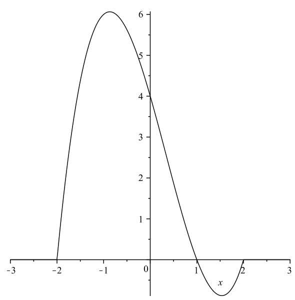

Figure 6.13: $y(x,0)$, Problem 17.

and

$$\tilde{y}(x,t) = \frac{1}{2}(\tilde{f}(x-ct) + \tilde{f}(x+ct)) + \frac{1}{2c}\tilde{g}(\xi)\,d\xi.$$

Then

$$\begin{aligned}
&y(x,t) - \tilde{y}(x,t) \\
&= \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi \\
&- \frac{1}{2}\left(\tilde{f}(x-ct) + \tilde{f}(x+ct)\right) - \frac{1}{2}\int_{x-ct}^{x+ct} \tilde{g}(\xi)\,d\xi \\
&= \frac{1}{2}(f(x-ct) - \tilde{f}(x-ct)) + \frac{1}{2}(f(x+ct) - \tilde{f}(x+ct)) \\
&+ \frac{1}{2c}\int_{x-ct}^{x+ct} (g(\xi) - \tilde{g}(\xi))\,d\xi.
\end{aligned}$$

Then

$$\begin{aligned}
&|y(x,t) - \tilde{y}(x,t)| \\
&\leq \frac{1}{2}|f(x-ct) - \tilde{f}(x-ct)| + \frac{1}{2}|f(x+ct) - \tilde{f}(x+ct)| \\
&+ \frac{1}{2c}\int_{x-ct}^{x+ct} |g(\xi) - \tilde{g}(\xi)|\,d\xi \\
&\leq \frac{1}{2}\epsilon_1 + \frac{1}{2}\epsilon_1 + \frac{1}{2c}\int_{x-ct}^{x+ct} \epsilon_2\,d\xi \\
&\leq \epsilon_1 + \frac{1}{2c}\epsilon_2((x+ct) - (x-ct)) \\
&= \epsilon_1 + \epsilon_2 t.
\end{aligned}$$

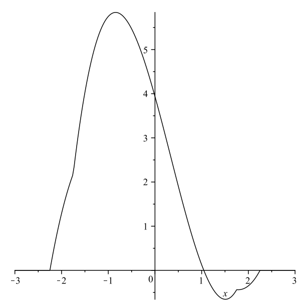

Figure 6.14: $y(x, 1/4)$, Problem 17.

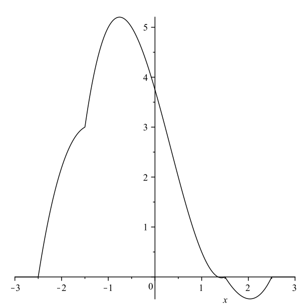

Figure 6.15: $y(x, 1/2)$, Problem 17.

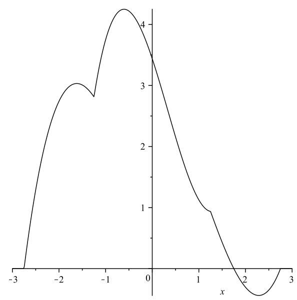

Figure 6.16: $y(x, 3/4)$, Problem 17.

6.4 The Wave Equation With a Forcing Term $K(x,t)$

1. Here $c = 4$, $f(x) = x$, $g(x) = e^{-x}$ and $K(x,t) = x + t$. The solution is

$$\begin{aligned}
y(x,t) &= \frac{1}{2}((x-4t)+(x+4t)) + \frac{1}{8}\int_{x-4t}^{x+4t} e^{-\xi}\,d\xi \\
&+ \frac{1}{8}\int_0^t \int_{x-4t+4T}^{x+4t-4T} (X+T)\,dX\,dT \\
&= x + \frac{1}{8}\left(e^{-x+4t} - e^{-x-4t}\right) \\
&+ \int_0^t (xt - xT + tT - T^2)\,dT \\
&= x + \frac{1}{8}e^{-x}\left(e^{4t} - e^{-4t}\right) + \frac{1}{6}t^3 + \frac{1}{2}xt^2.
\end{aligned}$$

We can also write this solution as

$$y(x,t) = x + \frac{1}{4}e^{-x}\sinh(4t) + \frac{1}{6}t^3 + \frac{1}{2}xt^2.$$

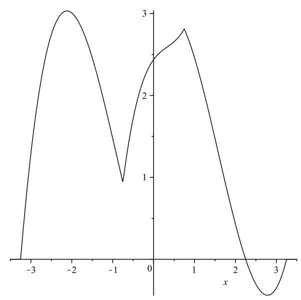

Figure 6.17: $y(x, 5/4)$, Problem 17.

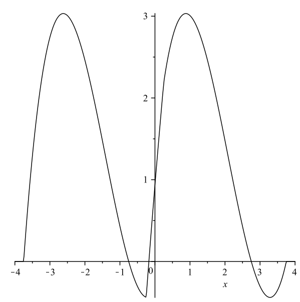

Figure 6.18: $y(x, 7/4)$, Problem 17.

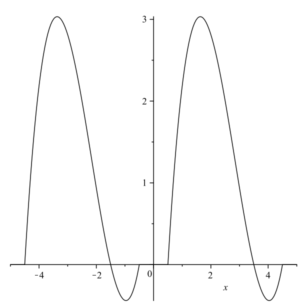

Figure 6.19: $y(x, 5/2)$, Problem 17.

3. The solution is

$$\begin{aligned}
y(x,t) &= \frac{1}{2}(f(x-8t)+f(x+8t)) + \frac{1}{16}\int_{x-8t}^{x+8t}\cos(2\xi)\,d\xi \\
&+ \frac{1}{16}\int_0^t \int_{x-8t+8T}^{x+8t-8T} XT^2\,dX\,dT \\
&= x^2+64t^2-x+\frac{1}{32}(\sin(-2x+16t)+\sin(2x+16t)) \\
&+ \int_0^t -xT^2(-t+T)\,dT \\
&= x^2+64t^2-x+\frac{1}{32}(\sin(-2x+16t)+\sin(2x+16t))+\frac{1}{32}xt^4.
\end{aligned}$$

5.

$$\begin{aligned}
y(x,t) &= \frac{1}{2}(\cosh(x-3t)+\cosh(x+3t)) + \frac{1}{6}\int_{x-3t}^{x+3t} d\xi \\
&+ \frac{1}{6}\int_0^t \int_{x-3t+3T}^{x+3t-3T} 3XT^3\,dX\,dT \\
&= \frac{1}{2}(\cosh(x-3t)+\cosh(x+3t)) + t + \int_0^t -3xT^3(T-t)\,dT \\
&= \frac{1}{2}(\cosh(x-3t)+\cosh(x+3t)) + t + \frac{3}{20}xt^5.
\end{aligned}$$

6.5 The Wave Equation in Higher Dimensions

For Problems 1–3 the solution has the form

$$z(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\sin(n\pi x/L)\sin(m\pi y/K)\cos(\alpha_{nm}\pi ct),$$

where

$$\alpha_{nm} = \sqrt{\frac{n^2}{L^2}+\frac{m^2}{K^2}}$$

and

$$a_{nm} = \frac{4}{LK}\int_0^L \int_0^K f(\xi,\eta)\sin(n\pi\xi/L)\sin(m\pi\eta/K)\,d\eta\,d\xi.$$

1. Because $f(x,y)=x^2y$ is a product of a function of x and a function of y, we can compute the coefficients as a product of integrals:

$$\begin{aligned}
a_{nm} &= \frac{1}{\pi^2}\int_0^{2\pi}\xi^2\sin(n\xi/2)\,d\xi\int_0^K \eta\sin(m\eta/2)\,d\eta \\
&= \frac{32(-1)^m}{mn^3\pi}\left(2(1-(-1)^n)+n^2\pi^2(-1)^n\right).
\end{aligned}$$

Further,

$$\alpha_{nm} = \frac{1}{2\pi}\sqrt{n^2+m^2}.$$

Because $c = 1$, the solution is

$$z(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{32(-1)^m}{mn^3\pi}\left(2(1-(-1)^n)\right.$$
$$\left. + n^2\pi^2(-1)^n\right)\sin(nx/2)\sin(my/2)\cos(\sqrt{n^2+m^2}t/2).$$

3. We have $c = 1, L = K = \pi$ and $f(x,y) = xe^y$. A routine integration yields

$$\begin{aligned} a_{nm} &= \frac{4}{\pi^2}\int_0^\pi \xi\sin(n\xi)\,d\xi\int_0^\pi e^\eta\sin(m\eta)\,d\eta \\ &= \frac{4(-1)^{n+1}m}{\pi n^2(m^2+1)}(1-e^\pi(-1)^m). \end{aligned}$$

Further,

$$\alpha_{nm} = \frac{1}{\pi}\sqrt{n^2+m^2}.$$

The solution is

$$z(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\sin(nx)\sin(my)\cos(2\sqrt{n^2+m^2}t).$$

5. Suppose $c = 3$, $L = K = \pi$, $f(x,y) = 0$ and $g(x,y) = xy$. Now

$$\alpha_{nm} = \frac{1}{\pi}\sqrt{n^2+m^2}.$$

Compute

$$b_{nm} = \frac{4}{3\sqrt{n^2+m^2}}\frac{(-1)^{n+m}}{nm}.$$

The solution is

$$z(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin(nx)\sin(mx)\cos(3\sqrt{n^2+m^2}t).$$

Chapter 7

Laplace's Equation

7.1 The Dirichlet Problem for a Rectangle

1. Substitute $u(x, y) = X(x)Y(y)$ into Laplace's equation to obtain

$$X'' + \lambda X = 0; X(0) = X(1) = 0$$

and

$$Y'' - \lambda Y = 0; Y(\pi) = 0.$$

Solutions for X are

$$\lambda = n^2\pi^2, X_n(x) = \sin(n\pi x).$$

With these values of λ, the problem for $Y(y)$ has solutions that are constant multiples of $\sinh(n\pi(\pi - y)$. To find a solution satisfying the boundary condition $u(x, 0) = \sin(\pi x)$, use a superposition

$$u(x, y) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \sinh(n\pi(\pi - y)).$$

We need

$$u(x, 0) = \sum_{n=1}^{\infty} a_n \sin(\pi x) \sinh(n\pi^2) = \sin(\pi x).$$

We can compute the Fourier coefficients of this sine expansion, or simply observe that we can take $a_n = 0$ for $n = 2, 3, \cdots$ and

$$a_1 = \frac{1}{\sinh(\pi^2)}.$$

The solution is

$$u(x, y) = \frac{1}{\sinh(\pi^2)}(\sin(\pi x) \sinh(\pi(\pi - y))).$$

3. After separating the variables and applying the boundary conditions, we find that the solution has the form

$$u(x, y) = \sum_{n=1}^{\infty} a_n \frac{\sinh(n\pi y)}{\sinh(4n\pi)} \sin(n\pi x).$$

The coefficients must be determined so that

$$u(x, 4) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) = x \cos(\pi x/2).$$

This is a Fourier sine expansion of $x\cos(\pi x/2)$ on $[0, 1]$. Choose the coefficients as

$$a_n = 2 \int_0^1 \xi \cos(\pi\xi/2) \sin(n\pi\xi) \, d\xi = \frac{32n(-1)^{n+1}}{\pi^2(4n^2 - 1)^2}.$$

These determine the solution.

5. There are nonhomogeneous boundary conditions on two sides of the rectangle, so write

$$u(x, y) = v(x, y) + w(x, y)$$

where

$$\nabla^2 v = 0; v(0, y) = v(\pi, y) = v(x, 0) = 0, v(x, \pi) = x \sin(\pi x)$$

and

$$\nabla^2 w = 0; w(x, 0) = w(x, \pi) = w(0, y) = 0, w(w, y) = \sin(y).$$

These are defined on $0 < x < 2, 0 < y < \pi$. Solve these problems independently.

First, separate variables in the problem for w to find that it has a solution of the form

$$w(x, y) = \sum_{n=1}^{\infty} b_n \sin(ny) \frac{\sinh(nx)}{\sinh(2n)}.$$

Observe that we can solve this problem for w by taking $b_1 = 1$ and all other $b_n = 0$, so

$$w(x, y) = \sin(y) \frac{\sinh(x)}{\sinh(2)}.$$

The problem for v has a solution of the form

$$v(x, y) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/2) \frac{\sinh(n\pi y/2)}{\sinh(n\pi^2/2)}.$$

We need

$$v(x, \pi) = x \sin(\pi x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/2).$$

This is a Fourier sine expansion of $x\sin(\pi x)$ on $[0,2]$, so choose

$$\begin{aligned} b_n &= \int_0^2 \xi \sin(n\xi)\sin(n\pi\xi/2)\,d\xi \\ &= \begin{cases} \frac{16n}{\pi^2((n^2-4)^2)}((-1)^n-1) & \text{for } n=1,3,4,5,\cdots, \\ 1 & \text{for } n=2. \end{cases} \end{aligned}$$

Then

$$\begin{aligned} v(x,y) &= \sin(\pi x)\frac{\sinh(\pi y)}{\sinh(\pi^2)} \\ &\quad + \frac{10}{\pi^2}\sum_{n=1,n\neq 2}^{\infty} \frac{n}{(n^2-4)^2}((-1)^n-1)\sin(n\pi x/2)\frac{\sinh(n\pi y/2)}{\sinh(n\pi^2/2)}. \end{aligned}$$

7. Separation of variables and the zero boundary conditions on $x=0, x=a$ and $y=0$ yield a general form of the solution:

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin((2n-1)\pi x/2a)\frac{\sinh((2n-1)\pi y/2a)}{\sinh((2n-1)\pi b/2a)}.$$

Now we need

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin((2n-1)\pi x/2a).$$

This leads us to choose

$$a_n = \frac{2}{a}\int_0^a f(\xi)\sin((2n-1)\pi\xi/2a)\,d\xi.$$

9. Write the solution as $u(x,y) = v(x,y) + w(x,y)$, where v is the solution of the problem

$$\nabla^2 v = 0, v(x,0) = v(x,1) = v(4,y) = 0, v(0,y) = \sin(\pi y),$$

and w is the solution of

$$\nabla^2(w) = 0, w(x,0) = w(x,1) = w(0,y) = 0, w(4,y) = y(1-y).$$

These problems are defined on $0 \leq x \leq 4, 0 \leq y \leq 1$. A separation of variables yields a general form of the solution of the problem for v:

$$v(x,y) = \sum_{n=1}^{\infty} a_n \sin(n\pi y)\frac{\sinh(n\pi(4-x))}{\sinh(4n\pi)}.$$

We need

$$v(0,y) = \sin(\pi y) = \sum_{n=1}^{\infty} a_n \sin(n\pi y),$$

so by observation we can let $a_1 = 1$ and $a_n = 0$ for $n = 2, 3, \cdots$. Then

$$v(x, y) = \sin(\pi y)\frac{\sinh(\pi(4-x))}{\sinh(4\pi)}.$$

Another separation of variables leads to a general form of the solution for w:

$$w(x, y) = \sum_{n=1}^{\infty} b_n \sin(n\pi y)\sinh(n\pi x).$$

Then

$$w(4, y) = y(1-y) = \sum_{n=1}^{\infty} b_n \sinh(4n\pi)\sin(n\pi y),$$

so

$$\begin{aligned} b_n &= \frac{2}{\sinh(4n\pi)}\int_0^1 \xi(1-\xi)\sin(n\pi\xi)\,d\xi \\ &= \frac{4(1-(-1)^n)}{n^3\pi^3\sinh(4n\pi)}. \end{aligned}$$

7.2 The Dirichlet Problem for a Disk

For each of Problems 1–8, a solution

$$u(r, \theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n [a_n\cos(n\theta) + b_n\sin(n\theta)],$$

where

$$a_n = \frac{1}{\pi R^n}\int_{-\pi}^{\pi} f(\xi)\cos(n\xi)\,d\xi$$

for $n = 0, 1, 2, \cdots$ and

$$b_n = \frac{1}{\pi R^n}\int_{-\pi}^{\pi} f(\xi)\sin(n\xi)\,d\xi$$

for $n = 1, 2, \cdots$.

1. We can see by observation that $u(r, \theta) = 1$ is a solution. This can also be obtained the long way by carrying out the integrations, obtaining $a_0 = 2$ and $a_n = b_n = 0$ for $n = 1, 2, \cdots$.

3. Calculate

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(\xi^2 - \xi)\,d\xi = \frac{2\pi^2}{3},$$

$$a_n = \frac{1}{2^n\pi}\int_{-\pi}^{\pi}(\xi^2 - \xi)\cos(n\xi)\,d\xi = \frac{4(-1)^n}{n^2 2^n}$$

and

$$b_n = \frac{1}{2^n\pi}\int_{-\pi}^{\pi}(\xi^2-\xi)\sin(n\xi)\,d\xi = \frac{2(-1)^n}{n2^n}.$$

The solution is

$$u(r,\theta) = \frac{\pi^2}{3} + 2\sum_{n=1}^{\infty}\left(\frac{r}{2}\right)^n \frac{(-1)^n}{n^2}[2\cos(n\theta)+n\sin(n\theta)].$$

5. The solution is

$$u(r,\theta) = \frac{\sinh(\pi)}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\left(\frac{r}{4}\right)^n \sinh(\pi)[\cos(n\theta)+n\sin(n\theta)].$$

7. After the integrations, we obtain the solution

$$u(r,\theta) = 1 - \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n^2}\left(\frac{r}{8}\right)^n\cos(n\theta).$$

9. Let $U(r,\theta) = u(r\cos(\theta), r\sin(\theta)$. The problem given in rectangular coordinates converts to the following problem in polar coordinates:

$$\nabla^2 U(r,\theta) = 0 \text{ for } 0 \le r < 4, U(4,\theta) = 16\cos^2(\theta).$$

If we write $16\cos^2(\theta) = 8(1+\cos(2\theta)$, we can recognize by inspection that

$$\frac{1}{2}a_0 = 8, a_2(4^2) = 8,$$

and all other $a_n = 0$. The solution in polar coordinates is

$$U(r,\theta) = 8 + 8\left(\frac{r}{4}\right)^2\cos(2\theta).$$

Because the original problem was posed in rectangular coordinates, convert this to rectangular coordinates by using $x = r\cos(\theta), y = r\sin(\theta)$, and the identity $\cos(2\theta) = 2\cos^2(\theta) - 1$, to obtain

$$u(x,y) = 8 + \frac{1}{2}(x^2 - y^2).$$

11. In polar coordinates this problem is

$$\nabla^2 U(r,\theta) = 0 \text{ for } 0 \le r < 2, U(2,\theta) = 4(\cos^2(\theta) - \sin^2(\theta)) = 4\cos(2\theta).$$

Identify $a_2 2^2 = 4$, with all other coefficients zero, so

$$U(r,\theta) = r^2\cos(2\theta).$$

In rectangular coordinates the solution is

$$u(x,y) = x^2 - y^2.$$

7.3 The Poisson Integral Formula

In Problems 1–4 the idea is to use the Poisson integral formula to write the requested solution value as an integral which can be approximated by a numerical technique. This assumes the availability of software that will do this.

For Problem 1, the Poisson integrals are given for each approximate value calculated. For Problems 2, 3 and 4, we give just the approximate values.

1. With $R = 1$ and $f(\theta) = \theta$, the solution is

$$U(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{pi} \frac{1-r^2}{1+r^2-2r\cos(\xi-\theta)}\xi\, d\xi.$$

Then

$$U(1/2,\pi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{3\xi}{5-4\cos(\xi-\pi)}\, d\xi = 0.$$

Next,

$$U(3/4,\pi/3) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{7\xi}{25-14\cos(\xi-\pi/3)}\, d\xi \approx 0.8826128645.$$

And

$$U(1/5,\pi/4) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{24\xi}{26-10\cos(\xi-\pi/4)}\, d\xi \approx 0.2465422.$$

3. $U(4,\pi) \approx -16.4654$, $U(12,\pi/6) \approx 0.0694$, $U(8,\pi/4) \approx 1.5281$

7.4 The Dirichlet Problem for Unbounded Regions

1. If we put $f(\xi) = K$ in equation (7.7), we get the solution

$$\begin{aligned} u(x,y) &= \frac{Ky}{\pi}\int_{-\infty}^{\infty} \frac{1}{y^2+(\xi-x)^2}\, d\xi \\ &= \frac{Ky}{\pi}\lim_{L\to\infty}\int_{-L}^{L} \frac{1}{y^2+(\xi-x)^2}\, d\xi \\ &= \frac{K}{\pi}\lim_{L\to\infty} \arctan\left(\frac{L-x}{y}\right) - \arctan\left(\frac{-L-x}{y}\right) \\ &= \frac{K}{\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = K. \end{aligned}$$

Or, we can avoid this computation by observing that $u(x,y) = K$ is harmonic on the entire plane, and equals K on the real line (the boundary of the upper half-plane). Therefore the solution is $u(x,y) = K$.

3. By equation (7.7), the solution is

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\xi}{y^2+(\xi-x)^2}\,d\xi.$$

5. Suppose $u(x,y)$ is harmonic on the upper half-plane and $u(x,0)=f(x)$. Then the function $v(x,y)$ defined by $v(x,y)=u(x,-y)$ on the lower half-plane is harmonic, and $v(x,0)=f(x)$. But we know an integral formula for $u(x,y)$. Therefore the problem for the lower half-plane has the solution

$$v(x,y) = u(x,-y) = -\frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)}{y^2+(\xi-x)^2}\,d\xi.$$

for all x and for $y<0$.

7. The boundary of the right quarter-plane consists of the nonnegative horizontal and vertical axes. Define two Dirichlet problems, in each of which boundary data is nonzero on just one part of the boundary:

   Problem 1 $\nabla^2 v = 0$ for $x>0, y>0$ and $v(x,0)=f(x), v(0,y)=0$, and

   Problem 2 $\nabla^2 w = 0$ for $x>0, y>0$ and $w(x,0)=0, w(0,y)=g(y)$.

   Both problems can be solved by separation of variables and Fourier integrals, obtaining

$$v(x,y) = \frac{2}{\pi}\int_0^{\infty}\left(\int_0^{\infty} f(\xi)\sin(\omega\xi)\,d\xi\right)\sin(\omega x)e^{-\omega y}\,d\omega$$

   and

$$w(x,y) = \frac{2}{\pi}\int_0^{\infty}\left(\int_0^{\infty} g(\eta)\sin(\omega\eta)\,d\eta\right)\sin(\omega y)e^{-\omega x}\,d\omega.$$

   The solution of the original problem is $u(x,y)=v(x,y)+w(x,y)$.

9.

$$\begin{aligned} u(x,y) &= \frac{Ay}{\pi}\int_4^8 \frac{1}{y^2+(\xi-x)^2}\,d\xi \\ &= \frac{A}{\pi}\left[\arctan\left(\frac{x-4}{y}\right) - \arctan\left(\frac{x-8}{y}\right)\right] \end{aligned}$$

11. Using the results of Problem 7, the solution is

$$u(x,y) = \int_0^{\infty} \left[b_\omega \sin(\omega x)e^{-\omega y} + B_\omega \sin(\omega y)e^{-\omega x}\right]d\omega,$$

where

$$\begin{aligned} b_\omega &= \frac{2}{\pi}\int_0^{\infty} \xi\sin(\omega\xi)\,d\xi \\ &= \frac{2\sin(\pi\omega) - 2\omega\pi\cos(\pi\omega)}{\pi\omega^2} \end{aligned}$$

and

$$\begin{aligned} B_\omega &= \frac{2}{\pi}\int_0^\infty \eta^2 \sin(\omega\eta)\, d\eta \\ &= \frac{4(\cos(\pi\omega)-1) - 2\omega^2\pi^2\cos(\pi\omega) + 4\pi\sin(\pi\omega)}{\pi\omega^3}. \end{aligned}$$

7.5 A Dirichlet Problem in 3 Dimensions

1. Let $u(x, y, z) = X(x)Y(y)Z(z)$ to separate variables, obtaining first that

$$X'' + \lambda X = 0; X(0) = X(1) = 0$$

and

$$Y'' + \mu Y - 0; Y(0) = Y(1) = 0.$$

Then

$$\lambda_n = n^2\pi^2, X_n(x) = \sin(n\pi x)$$

and

$$\mu_m = m^2\pi^2, Y_m(y) = \sin(m\pi y).$$

Further,

$$Z'' - (n^2 + m^2)\pi^2 Z = 0; Z(0) = 0.$$

This leads to functions

$$u_{nm}(x, y, z) = c_{nm}\sin(n\pi x)\sin(m\pi y)\sinh(\alpha_{nm}\pi z),$$

where $\alpha_{nm} = \sqrt{n + m^2}$. To satisfy the condition $u(x, y, 1) = xy$, use a superposition

$$u(x, y, z) = \sum_{n=1}^\infty \sum_{m=1}^\infty c_{nm}\sin(n\pi x)\sin(m\pi y)\sinh(\alpha_{nm}\pi z).$$

We must choose the coefficients so that

$$u(x, y, 1) = xy = \sum_{n=1}^\infty \sum_{m=1}^\infty \sin(n\pi x)\sin(m\pi y)\sinh(\alpha_{nm}\pi).$$

This is a double Fourier series on the square $0 \le x \le 1$, $0 \le y \le 1$, and we know from experience with the heat and wave equations that

$$\begin{aligned} c_{nm} &= \frac{4}{\sinh(\alpha_{nm}\pi)}\int_0^1 \xi\sinh(n\pi\xi)\, d\xi \int_0^1 \eta\sin(m\pi\eta)\, d\eta \\ &= \frac{4(-1)^{n+m}}{nm\pi^2\sinh(\alpha_{nm}\pi)}. \end{aligned}$$

3. The solution is the sum of the solutions of the following two problems:

$$\nabla^2 w = 0,$$
$$w(0, y, z) = w(1, y, z) = w(x, 0, z) = w(x, 2\pi, z) = w(x, y, 0) = 0,$$
$$w(x, y, \pi) = 1$$

and

$$\nabla^2 v = 0,$$
$$v(0, y, z) = v(1, y, z) = v(x, y, 0) = v(x, y, \pi) = v(x, 0, z) = 0,$$
$$v(x, 2\pi, z) = xz^2.$$

Each of these problems is solved by a separation of variables. For the first, we obtain

$$w(x, yz) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \sin(n\pi x) \sin(my/2) \sinh(\sqrt{4n^2\pi^2 + m^2} z/2),$$

in which

$$\begin{aligned} a_{nm} &= \frac{1}{\sinh(\sqrt{4n^2\pi^2 + m^2}\pi/2)} \int_0^1 2\sin(n\pi\xi)\, d\xi \int_0^{2\pi} \frac{1}{\pi} \sin(n\pi\eta)\, d\eta \\ &= \frac{1}{\sinh(\sqrt{n^2\pi^2 + m^2}\pi/2)} \left(\frac{1-(-1)^n}{n\pi}\right) \left(\frac{1-(-1)^m}{m\pi}\right). \end{aligned}$$

For the second problem, obtain

$$v(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{nm} \sin(n\pi x) \sin(mz) \sinh(\sqrt{n^2\pi^2 + m^2} y),$$

in which

$$\begin{aligned} b_{nm} &= \frac{4}{\pi^2 \sinh(\sqrt{n^2\pi^2 + m^2} 2\pi)} \int_0^1 \xi \sin(n\pi\xi)\, d\xi \int_0^{\pi} \tau^2 \sin(m\tau)\, d\tau) \\ &= \frac{4}{\pi^2 \sinh(\sqrt{n^2\pi^2 + m^2} 2\pi)} \left(\frac{(-1)^{n+1}}{n\pi}\right) \left(\frac{2 - 2(-1)^m + m^2\pi^2(-1)^m}{m^3}\right). \end{aligned}$$

The original problem has solution

$$u(x, y, z) = w(x, y, z) + v(x, y, z).$$

7.6 The Neumann Problem

1. First,

$$\int_0^1 4\cos(\pi x)\, dx = 0,$$

so this problem may have a solution. (If this integral were nonzero, we could conclude that the problem has no solution).

A separation of variables, making use of the three homogeneous boundary conditions, leads to the problems

$$X'' + \lambda X = 0; X'(0) = X'(1) = 0$$

and

$$Y'' - \lambda Y = 0; Y'(1) = 0.$$

These have solutions of the form

$$\lambda_n = n^2\pi^2, X_n(x) = \cos(n\pi y)$$

and

$$Y_n(x) = \cosh(n\pi(1-y)).$$

Thus attempt a solution of the Neumann problem of the form

$$u(x,y) = c_0 + \sum_{n=1}^{\infty} c_n \cosh(n\pi(1-y))\cos(n\pi y).$$

The condition $u_y(0) = 4\cos(\pi x)$ requires that

$$\sum_{n=1}^{\infty} -c_n n\pi \sinh(n\pi)\cos(n\pi x) = 4\cos(\pi x).$$

This is satisfied if we put $c_n = 0$ for $n = 2, 3, 4, \cdots$, and choose c_1 so that

$$-c_1\pi\sinh(1)\cos(\pi x) = 4\cos(\pi x).$$

Therefore $-c_1\pi\sinh(\pi) = 4$, and

$$c_1 = -\frac{4}{\pi\sinh(\pi)}.$$

The solution is

$$u(x,y) = c_0 - \frac{4}{\pi\sinh(\pi)}\cosh(\pi(1-y))\cos(\pi x).$$

Here c_0 is an arbitrary constant, so this solution is not unique.

3. A solution may exist because $\int_0^\pi \cos(3x)\,dx = 0$. From the zero boundary conditions on edges $x = 0$ and $x = \pi$, separation of variables yields a solution of the form

$$u(x,y) = c_0 + \sum_{n=1}^{\infty}[c_n\cosh(ny) + d_n\cosh(n(\pi-y))]\cos(nx).$$

Now
$$\frac{\partial u}{\partial y}(x,0) = \cos(3x) = \sum_{n=1}^{\infty} -nd_n \sinh(n\pi)\cos(nx)$$
so
$$d_3 = -\frac{1}{3\sinh(3\pi)}$$
and $d_n = 0$ if $n \neq 3$. Next, the boundary condition at $y = \pi$ gives us
$$\frac{\partial u}{\partial u}(x,\pi) = 6x - 3\pi = \sum_{n=1}^{\infty} nc_n \sinh(n\pi)\cos(nx).$$
Then
$$\begin{aligned} c_n &= \frac{1}{n\sinh(n\pi)}\frac{2}{\pi}\int_0^{\pi}(6x-3\pi)\cos(nx)\,dx \\ &= \frac{1}{n\sinh(n\pi)}\frac{12}{n^2\pi}((-1)^n - 1). \end{aligned}$$
The solution is
$$\begin{aligned} u(x,y) = c_0 &- \frac{\cosh(3(\pi-y))}{3\sinh(3\pi)}\cos(3x) \\ &+ \sum_{n=1}^{\infty}\frac{12((-1)^n-1)}{n^3\pi\sinh(n\pi)}\cosh(ny)\cos(nx). \end{aligned}$$

5. With $u(x,y) = X(x)Y(y)$, we obtain:
$$X'' - \lambda X = 0$$
and
$$Y'' + \lambda Y = 0; Y(0) = Y(1) = 0.$$
Then
$$Y_n(y) = \sin(n\pi y) \text{ and } X_n(x) = c_n\cosh(n\pi x) + d_n\cosh(n\pi(1-x))$$
for $n = 1, 2, \cdots$. Look for a solution of the form
$$u(x,y) = \sum_{n=1}^{\infty}[c_n\cosh(n\pi x) + d_n\cosh(n\pi(1-x))]\sin(n\pi y).$$
To solve for the constants, use the other two boundary conditions. First,
$$\frac{\partial u}{\partial x}(1,y) = 0 = \sum_{n=1}^{\infty} n\pi c_n\sinh(n\pi)\sin(n\pi y)$$

so we each $c_n = 0$. Next,

$$\frac{\partial u}{\partial x}(0, y) = 3y^2 - 2y = \sum_{n=1}^{\infty} -n\pi d_n \sinh(n\pi)\sin(n\pi y).$$

Then

$$\begin{aligned} d_n &= \frac{-2}{n\pi\sinh(n\pi)}\int_0^1 (3\eta^2 - 2\eta)\sin(n\pi\eta)\,d\eta \\ &= \frac{2}{n^4\pi^4\sinh(n\pi)}[n^2\pi^2(-1)^n + 6(1-(-1)^n)] \end{aligned}$$

for $n = 1, 2, \cdots$. The solution is

$$u(x, y) =$$
$$\sum_{n=1}^{\infty} \frac{2}{n^4\pi^4\sinh(n\pi)}[n^2\pi^2(-1)^n + 6(1-(-1)^n)]\cosh(n\pi(1-x))\sin(n\pi y).$$

7. First check that $\int_{-\pi}^{\pi}\cos(2\theta)\,d\theta = 0$, a necessary condition for a solution to exist. A solution must have the form

$$u(r, \theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n\cos(n\theta) + b_n\sin(n\theta)].$$

From the boundary condition at $r = R$, we have

$$\begin{aligned} \frac{\partial u}{\partial r}(R, \theta) &= \cos(2\theta) \\ &= \sum_{n=1}^{\infty}[na_nR^{n-1}\cos(n\theta) + nb_nR^{n-1}\sin(n\theta)]. \end{aligned}$$

As in the preceding problem, compare coefficients on both sides of this equation to choose each $b_n = 0$ and $a_n = 0$ except for $n = 2$. Further, $2a_2R = 1$. The solution is

$$u(r, \theta) = \frac{1}{2}a_0 + \frac{R}{2}\left(\frac{r}{R}\right)^2\cos(2\theta).$$

9. Because

$$\int_{-\infty}^{\infty} e^{-|\xi|}\sin(\xi)\,d\xi = 0,$$

a necessary condition for a solution to exist is satisfied. The solution is

$$u(x, y) = \frac{1}{2\pi}\int -\infty^{\infty}\ln(y^2 + (\xi - x)^2)e^{-|\xi|}\sin(\xi)\,d\xi.$$

11. Problem 7, Section 7.4 requested a solution of the Dirichlet problem for the right quarter-plane. Using this, we are led to attempt a solution for the Neumann problem for the right quarter-plane of the form

$$u(x,y) = \int_0^\infty a_\omega \cos(\omega x) e^{-\omega y}\, d\omega.$$

Now

$$\frac{\partial u}{\partial y}(x,0) = \int_0^\infty -\omega a_\omega \cos(\omega x)\, d\omega.$$

This tells us to choose

$$a_\omega = -\frac{2}{\pi\omega}\int_0^\infty f(\xi)\cos(\omega\xi)\, d\xi.$$

7.7 Poisson's Equation

1. Write $u(x,y) = v(x,y) + w(x,y)$, where v is the solution of the Dirichlet problem

$$\begin{aligned} \nabla^2 v &= 0 \text{ for } 0 < x < 1, 0 < y < 1,\\ v(x,0) &= v(x,1) = 0,\\ v(1,y) &= 0,\\ v(0,y) &= y \end{aligned}$$

and w is the solution of the problem

$$\begin{aligned} \nabla^2 w &= 0 \text{ for } 0 < x < 1, 0 < y < 1,\\ w(0,y) &= w(1,y) = w(x,0) = w(x,1) = 0,\\ w(x,y) &= xy \text{ for } 0 < x < 1, 0 < y < 1. \end{aligned}$$

For the first problem, for $v(x,y)$, separate variables to obtain the solution:

$$v(x,y) = \sum_{n=1}^\infty a_n \sin(n\pi(1-x))\sin(n\pi y),$$

where

$$a_n = \frac{2(-1)^{n+1}}{n\pi\sinh(n\pi)}.$$

The problem for $w(x,y)$ has the solution

$$w(x,y) = \sum_{n=1}^\infty\sum_{m=1}^\infty k_{nm}\sin(n\pi x)\sin(m\pi y),$$

where

$$\begin{aligned} k_{nm} &= \frac{-4}{\pi^2(n^2+m^2)} \int_0^1 \xi \sin(n\pi\xi \, d\xi \int_0^1 \eta \sin(m\pi\eta) \, d\eta \\ &= \frac{4(-1)^{n+m+1}}{(n^2+m^2)nm\pi^4}. \end{aligned}$$

3. Split the problem into two problems, as we have been doing. However, the first problem (see Figure 7.6) must itself be broken up into two problems, in the first of which $v(0, y) = 1$ and $v(\pi, y) = 0$, and in the second of which $v(0, y) = 0$ and $v(\pi, y) = 0$. Applying straightforward separation of variables to these problems, we obtain

$$v_1(x,y) = \sum_{n=1}^{\infty} \frac{2}{n\pi \sinh(n\pi)} (1 - (-1)^n) \sin(my) \sinh(n(\pi - x))$$

for the Dirichlet problem with $v(0, y) = 1$ and $v(\pi, y) = 0$. If $v(0, y) = 0$ and $v(\pi, y) = y$, we obtain

$$v_2(x,y) = \sum_{n=1}^{\infty} \frac{2}{n \sinh(n\pi)} (-1)^{n+1} \sin(ny) \sinh(nx).$$

For the problem for w in Figure 7.6, we have

$$w(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} k_{nm} \sin(nx) \sin(my),$$

where

$$\begin{aligned} &k_{nm} = \\ &-\frac{4}{\pi^2} n^2\pi^2 + m^2\pi^2 \int_0^{\pi} \int_0^{\pi} \xi^2 \sin(n\xi) \, d\xi \int_0^{\pi} \eta^2 \sin(m\eta) \, d\eta \\ &= \frac{4}{\pi^2(n^2+m^2)} (2 - 2(-1)^n + n^2\pi^2(-1)^n)(-2 + 2(-1)^m - m^2\pi^2(-1)^m). \end{aligned}$$

Chapter 8

Special Functions and Applications

8.1 Legendre Polynomials

For Problems 1–4 and 6, graphs of the function and the sixth partial sum of its Fourier-Legendre expansion on$[-1, 1]$ appear nearly indistinguishable within the scale of the graph. The “most“ functions many terms of this expansion are needed to achieve a good fit between the partial sum and the function. This is seen in Problem 5, where the sixth partial sum is a poor fit to the function, while the fiftieth partial sum is much closer (though still a poor fit).

1. The coefficients are

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} \sin(\pi x/2) P_n(x)\, dx.$$

Carrying out these integrations, we obtain

$$\begin{aligned} &c_0 = c_2 = c_4 = 0, c_1 = \frac{12}{\pi^2}, \\ &c_3 = \frac{168(\pi^2 - 10)}{\pi^4}, c_5 = \frac{660(\pi^4 - 112\pi^2 + 1008)}{\pi^6}. \end{aligned}$$

Figure 1 shows a graph of $f(x)$ and $\sum_{n=0}^{5} c_n P_n(x)$.

3. The coefficients are

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} \sin^2(x) P_n(x)\, dx.$$

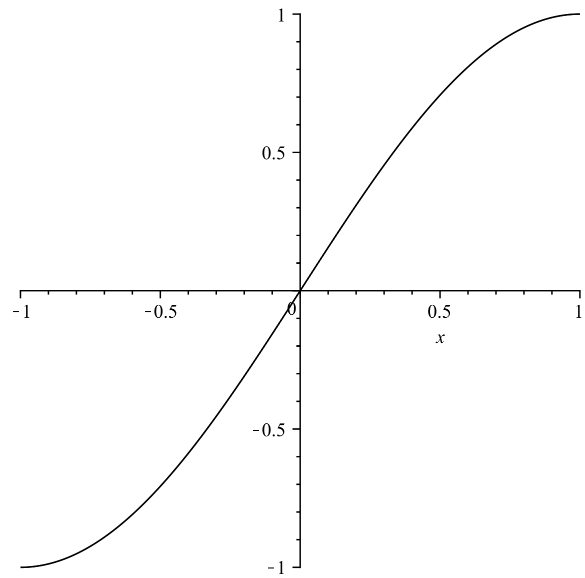

Figure 8.1: Graph of $\sin(\pi x/2)$ and the sixth partial sum of its Fourier-Legendre expansion.

The first six are

$$c_0 = -\frac{1}{2}\sin(1)\cos(1) + \frac{1}{2}, c_1 = c_3 = c_5 = 0,$$
$$c_2 = -\frac{5}{8}\sin(1)\cos(1) + \frac{15}{8} - \frac{15}{4}\cos^2(1),$$
$$c_4 = -\frac{585}{32} + \frac{585}{16}\cos^2(1) + \frac{531}{32}\sin(1)\cos(1).$$

Figure 8.2 shows the function and the sixth partial sum of this Fourier-Legendre expansion.

5. The coefficients are

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx.$$

The first six coefficients are

$$c_0 = c_2 = c_4 = 0, c_1 = \frac{3}{2}, c_3 = -\frac{7}{8}, c_5 = \frac{11}{16}.$$

Figure 8.3 shows a graph of the function and this partial sum. For this function the sixth partial sum does not fit the function well at all on $[-1, 1]$. Figure 8.4 shows the fiftieth partial sum, a better fit to the function.

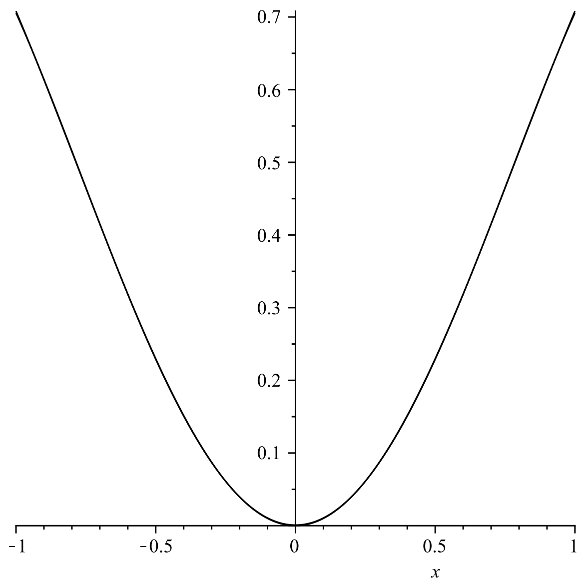

Figure 8.2: Graph of $\sin^2(x)$ and the sixth partial sum of its Fourier-Legendre expansion.

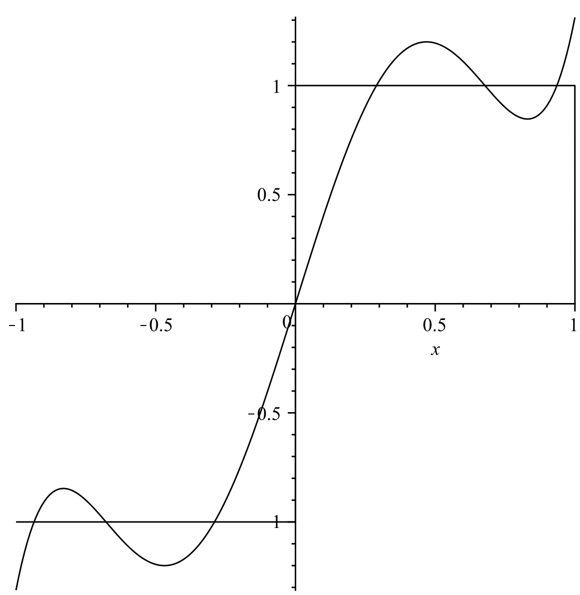

Figure 8.3: Graph of $f(x)$ and the sixth partial sum of its Fourier-Legendre expansion.

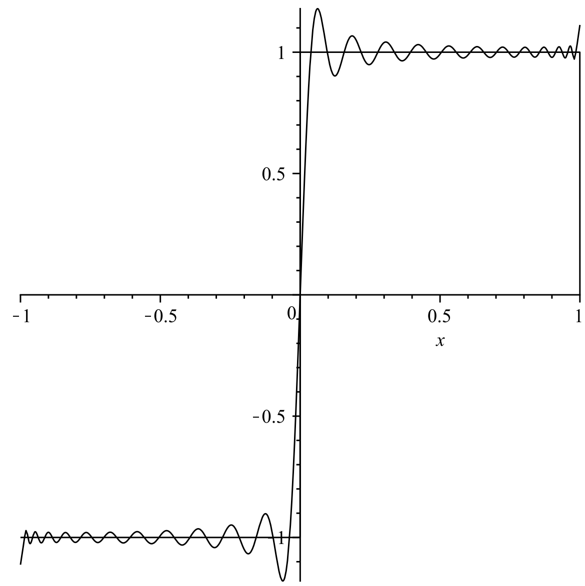

Figure 8.4: Graph of $f(x)$ and the fiftieth partial sum of its Fourier-Legendre expansion.

7. For $n = 7$, $[n/2] = [7/2] = 3$, so

$$\begin{aligned} P_7 &= \sum_{k=0}^{3} (-1)^k \frac{(14-2k)!}{2^7 k!(7-k)!(7-2k)!} x^{n-2k} \\ &= \frac{429}{16}x^7 - \frac{693}{16}x^5 + \frac{315}{16}x^3 - \frac{35}{16}x. \end{aligned}$$

For $n = 8$, $[n/2] = [4] = 4$ and

$$\begin{aligned} P_8(x) &= \sum_{k=0}^{4} (-1)^k \frac{(16-2k)!}{2^8 k!(8-k)!(8-2k)!} x^{8-2k} \\ &= \frac{6435}{128}x^8 - \frac{3003}{32}x^6 + \frac{3465}{64}x^4 \\ &- \frac{315}{33}x^2 + \frac{35}{128}. \end{aligned}$$

For $n = 9$, $[n/2] = [9/2] = 4$ and

$$\begin{aligned} P_9(x) &= \sum_{k=0}^{4} (-1)^k \frac{(18-2k)!}{2^9 k!(9-k)!(9-2k)!} x^{9-2k} \\ &= \frac{12155}{128} x^9 - \frac{6435}{32} x^7 + \frac{9009}{64} x^5 \\ &- \frac{1155}{32} x^3 + \frac{315}{128} x. \end{aligned}$$

For $n = 10$, $[n/2] = [5] = 5$ and

$$\begin{aligned} P_{10}(x) &= \sum_{k=0}^{5} (-1)^k \frac{(20-2k)!}{2^{10} k!(10-k)!(10-2k)} x^{10-2k} \\ &= \frac{46189}{256} x^{10} - \frac{109395}{256} x^8 + \frac{45045}{128} x^6 \\ &- \frac{15015}{128} x^4 + \frac{3465}{256} x^2 - \frac{63}{256}. \end{aligned}$$

9. Let

$$Q_n(x) = \frac{1}{\pi} \int_0^\pi \left(x + \sqrt{x^2-1}\cos(\theta)\right)^n \, d\theta$$

for $n = 0, 1, 2, \cdots$. The strategy is to show that $Q_n(x)$ satisfies the same recurrence relation (8.7) that the Legendre polynomials do. From Problem 8, we also have that $Q_0(x) = P_0(x)$ and $Q_1(x) = P_1(x)$. Then the recurrence relation will give us $Q_2(x) = P_2(x)$, and then $Q_4(x) = P_4(x)$, and so on.

To show that $Q_n(x)$ satisfies equation (8.7), first substitute the integral for $Q_n(x)$ into this equation and rearrange terms to obtain

$$\begin{aligned} &\frac{1}{\pi} \int_0^\pi \left(-n(x^2-1)\sin^2(\theta) + \sqrt{x^2-1}\cos(\theta)[x + \sqrt{x^2-1}\cos(\theta)]\right) \\ &\times \left(x + \sqrt{x^2-1}\cos(\theta)\right)^{n-1} \, d\theta. \end{aligned}$$

Now integrate this by parts, with

$$u = \left(x + \sqrt{x^2-1}\cos(\theta)\right)^n$$

and

$$dv = \cos(\theta)\, d\theta$$

to obtain

$$\begin{aligned} &\frac{1}{\pi} \int_0^\pi \left(x + \sqrt{x^2-1}\cos(\theta)\right)^n \sqrt{x^2-1}\cos(\theta)\, d\theta \\ &= \frac{1}{\pi} \left(x + \sqrt{x^2-1}\cos(\theta)\right)^{n-1} n(x^2-1)\sin^2(\theta). \end{aligned}$$

Use this in the substitution of $Q_n(x)$ into equation (8.7) to show that $Q_n(x)$ satisfies this recurrence relation. This shows that $Q_n(x) = P_n(x)$.

11. Put $x = t = 1/2$ into the generating formula for the Legendre polynomials to get

$$\frac{1}{\sqrt{3/4}} = \sum_{n=0}^{\infty} P_n\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^n.$$

Then

$$\frac{2}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{1}{2^n} P_n\left(\frac{1}{2}\right).$$

Then

$$\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} P_n\left(\frac{1}{2}\right) = \frac{1}{\sqrt{3}}.$$

13. Apply the law of cosines to the triangle in the diagram to get

$$R^2 = r^2 + d^2 - 2rd\cos(\theta).$$

Then

$$\frac{R^2}{d^2} = 1 - 2\frac{r}{d}\cos(\theta) + \frac{r^2}{d^2}.$$

Then

$$\varphi(x,y,z) = \frac{1}{R} = \frac{1}{d}\frac{d}{R} = \frac{1}{d}\frac{1}{\sqrt{1 - 2\frac{r}{d}\cos(\theta) + \frac{r^2}{d^2}}}.$$

For the remainder of the problem, consider two cases on r/d. First, suppose $r/d < 1$, so $r < d$. Put $x = \cos(\theta)$ and $t = r/d$ in the generating function to obtain

$$\varphi(r) = \frac{1}{d}\sum_{n=0}^{\infty} P_n(\cos(\theta))\left(\frac{r}{d}\right)^n,$$

or

$$\varphi(r) = \sum_{n=0}^{\infty} \frac{1}{d^{n+1}} P_n(\cos(\theta)) r^n.$$

If $r/d > 1$, so $r > d$, now write

$$\frac{R^2}{r^2} = 1 - 2\frac{d}{r}\cos(\theta) + \frac{d^2}{r^2}.$$

Then

$$\frac{r}{R} = \frac{1}{\sqrt{1 - 2\frac{d}{r}\cos(\theta) + \frac{d^2}{r^2}}}.$$

Again, comparing this with the generating function, we have

$$\varphi(r) = \frac{1}{r}\sum_{n=0}^{\infty} P_n(\cos(\theta))\left(\frac{d}{r}\right)^n,$$

and this is equivalent to

$$\varphi(r) = \frac{1}{r}\sum_{n=0}^{\infty} d^n P_n(\cos(\theta))e^{-n}.$$

15. With $f(\varphi) = \sin(\varphi)$, we have

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} \sin(\arccos(\xi))P_n(\xi)\,d\xi$$

and

$$u(\rho,\varphi) = \sum_{n=0}^{\infty} c_n \left(\frac{\rho}{R}\right)^n P_n(\cos(\varphi)).$$

In computing the coefficients, use can be made of the identity

$$\sin(\arccos(\xi)) = \sqrt{1-\xi^2}.$$

With $R = 1$, and using the twenty-first partial sum of the solution, we obtain the approximations

$$u(1,\pi/4) \approx 0.707274, u(1,\pi/6) \approx 0.500761, u(1,\pi/8) \approx 0.382683.$$

17. With $f(\varphi) = 2 - \varphi^2$, let

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} (2 - \arccos^2(\xi))P_n(\xi)\,d\xi$$

and

$$u(\rho,\varphi) = \sum_{n=0}^{\infty} c_n \left(\frac{\rho}{R}\right)^n P_n(\cos(\varphi)).$$

With $R = 1$, use the twenty-first partial sum to approximate:

$$u(1,\pi/4) \approx 1.384743, u(1,\pi/6) \approx 1.725844, u(1,\pi/8) \approx 1.845787.$$

19. In spherical coordinates, the Dirichlet problem to be solved is:

$$u_{\rho\rho} + \frac{2}{\rho}u_{\varphi\varphi} + \frac{\cot(\varphi)}{\rho^2}u_\varphi = 0, R_1 < \rho < R_2, -\pi/2 \le \varphi \le \pi/2,$$
$$u(R_1,\varphi) = T, U(R_2,\varphi) = 0.$$

This problem can be solved by separation of variables. Let

$$u(\rho, \varphi) = F(\rho)\Phi(\varphi).$$

This results in:

$$F'' + \frac{2}{\rho}F' - \frac{\lambda}{\rho^2}F = 0$$

and

$$\Phi'' + \cot(\varphi)\Phi' + \lambda\Phi = 0.$$

The equation for $\Phi(\varphi)$ has the bounded solution

$$\Phi_n(\varphi) = P_n(\cos(\varphi)),$$

corresponding to an eigenvalue $\lambda_n = n(n+1)$ of Legendre's equation. For $n = 0, 1, 2, \cdots$, solutions for $F(\rho)$ are

$$F_n(\rho) = a_n\rho^n + b_n\rho^{-n-1}.$$

Attempt a superposition

$$u(\rho, \varphi) = \sum_{n=0}^{\infty}(a_n\rho^n + b_n\rho^{-n-1})P_n(\cos(\varphi)).$$

We require that

$$u(R_1, \varphi) = T = \sum_{n=0}^{\infty}(a_nR_1^n + b_nR_1^{-n-1})P_n(\cos(\varphi)).$$

And the condition at $\rho = R_2$ is that

$$r(R_2, \varphi) = 0 = \sum_{n=0}^{\infty}(a_nR_2^n + b_nR_2^{-n-1})P_n(\cos(\varphi)).$$

Recalling that $P_0(\cos(\varphi)) = 1$, these equations are satisfied if we choose the coefficients so that

$$a_0 + b_0R_1^{-1} = T, a_0 + b_0R_2^{-1} = 0$$

and, for $n = 1, 2, \cdots$, let $a_n = b_b = 0$. We should therefore let

$$a_0 = \frac{TR_1}{R_1 - R_2} \text{ and } b_0 = -\frac{TR_1R_2}{R_1 - R_2}.$$

The solution is

$$u(\rho, \varphi) = \frac{TR_1}{R_1 - R_2}\left[\frac{R_2}{\rho} - 1\right].$$

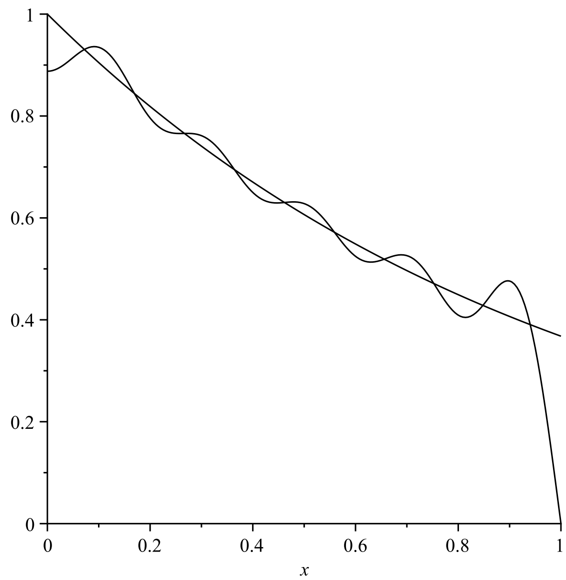

Figure 8.5: Graph of e^{-x} and the tenth partial sum of its Fourier-Bessel expansion.

8.2 Bessel Functions

1. With $f(x) = e^{-x}$, the expansion in terms of zero-order Bessel functions is

$$\sum_{n=1}^{\infty} c_n J_0(j_n x),$$

where j_n is the nth (in increasing order) positive zero of $J_0(x)$ and

$$c_n = \frac{2}{J_1^2(j_n)} \int_0^1 \xi e^{-\xi} J_0(j_n \xi)\, d\xi.$$

Figure 8.5 shows the function and the tenth partial sum of this expansion, and Figure 8.6 shows the twenty-fifth partial sum.

3. Let j_n be the nth positive zero of $J_1(x)$ and

$$c_n = \frac{2}{J_2^2(j_n)} \int_0^1 \xi^3 e^{-2\xi} J_1(j_n \xi)\, d\xi.$$

The Fourier-Bessel expansion is

$$\sum_{n=1}^{\infty} c_n J_1(j_n x).$$

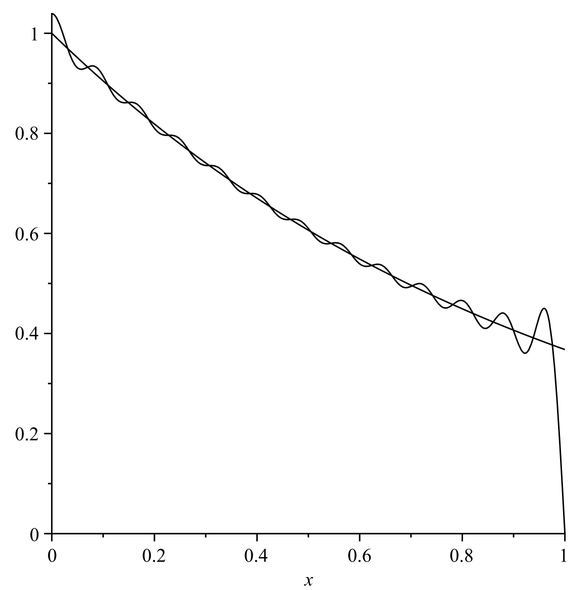

Figure 8.6: Graph of e^{-x} and the twenty-fifth partial sum of its Fourier-Legendre expansion.

Figure 8.7 shows the function and the twentieth partial sum of this series, while Figure 8.8 has the fortieth partial sum.

5. Let j_n be the nth positive zero of $J_4(x)$, and

$$c_n = \frac{2}{J_5^2(j_n)} \int_0^1 \xi \sin(3\xi) J_4(j_n \xi)\, d\xi.$$

The Fourier-Bessel expansion is

$$\sum_{n=1}^{\infty} c_n J_4(j_n x).$$

Figure 8.9 shows the function and the twentieth partial sum of this series, while Figure 8.10 has the fortieth partial sum.

7. A formal proof that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin(x)$$

can be made by using the Maclaurin expansion of $\sin(x)$ on the right side and manipulating the coefficients to obtain the series defining $J_{1/2}(x)$. We will be less formal here and essentially carry out this type of argument

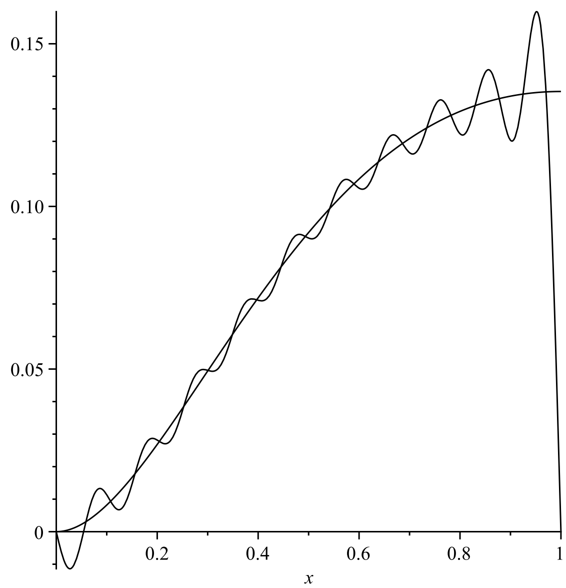

Figure 8.7: Graph of x^2e^{-2x} and the twentieth partial sum of its Fourier-Legendre expansion.

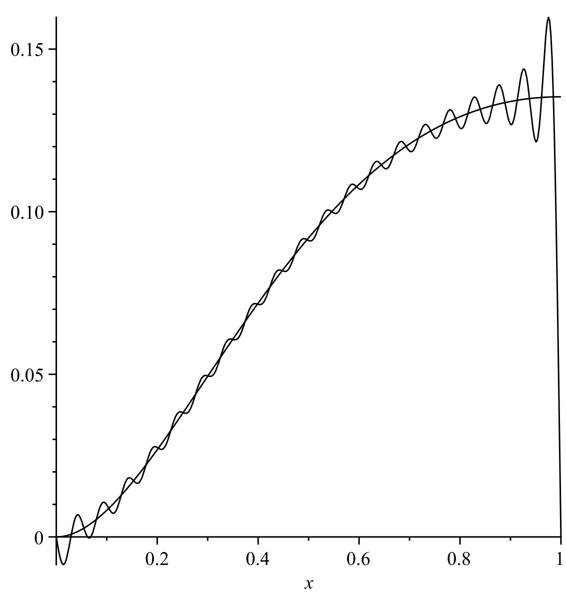

Figure 8.8: Graph of x^2e^{-2x} and the fortieth partial sum of its Fourier-Legendre expansion.

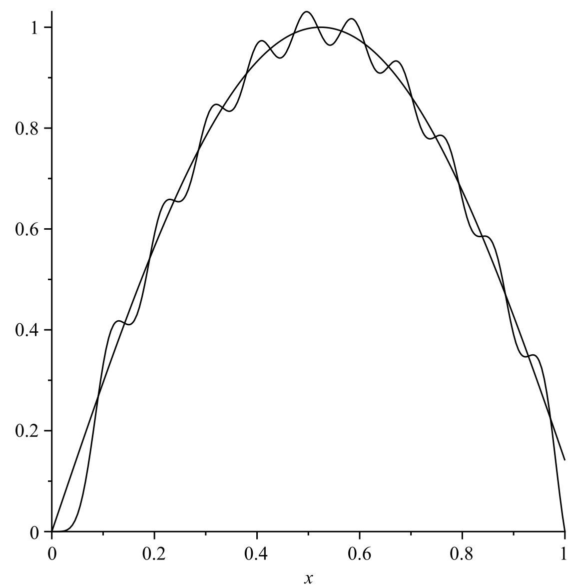

Figure 8.9: Graph of $\sin(3x)$ and the twentieth partial sum of its Fourier-Legendre expansion.

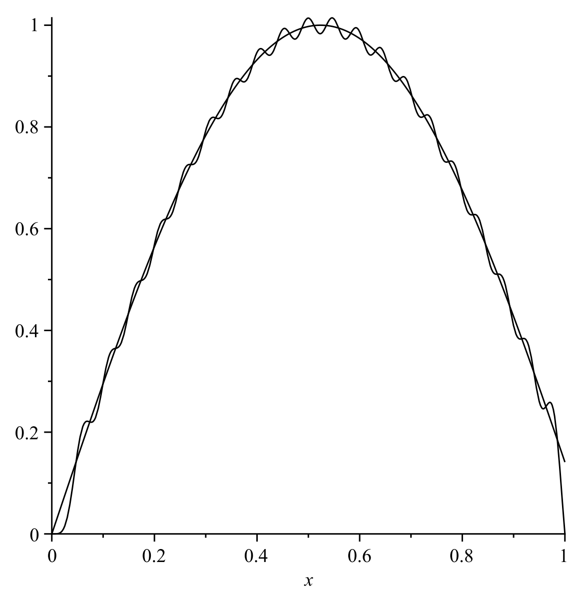

Figure 8.10: Graph of $\sin(3x)$ and the fortieth partial sum of its Fourier-Legendre expansion.

using a few terms of the series so that it is apparent what is happening. Begin with

$$\begin{aligned}
J_{1/2}(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1/2} n! \Gamma(n+1/2+1)} x^{2n+1/2} \\
&= \sqrt{\frac{x}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n! \Gamma(n+1/2+1)} x^{2n} \\
&= \sqrt{\frac{x}{2}} \left[\frac{1}{\Gamma(1/2+1)} - \frac{1}{2^2 \Gamma(1+1/2+1)} x^2 \right. \\
&\left. + \frac{1}{2^4 2! \Gamma(2+1/2+1)} - \frac{1}{2^6 3! \Gamma(3+1/2+1)} x^6 + \frac{1}{2^8 4! \Gamma(4+1/2+1)} x^8 + \cdots \right].
\end{aligned}$$

Now we need to know some values of the gamma function. First,

$$\Gamma(1/2) = \int_0^{\infty} t^{-1/2} e^{-t}\, dt.$$

Letting $t = u^2$, this is

$$\begin{aligned}
\Gamma(1/2) &= \int_0^{\infty} \frac{1}{u} e^{-u^2} 2u\, du \\
&= 2 \int_0^{\infty} e^{-u^2}\, du = \sqrt{\pi}.
\end{aligned}$$

Here we have used the well-known result that

$$\int_0^{\infty} e^{-u^2}\, du = \frac{1}{2}\sqrt{\pi},$$

which can be derived using double integrals or complex integration, and is widely used in probability and statistics. Then, using the factorial property of the gamma function, we can evaluate $\Gamma(n+1/2+1)$ for various values of n. In particular,

$$\Gamma\left(1+\frac{1}{2}+1\right) = \left(1+\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}\right) = \frac{3}{2}\frac{\sqrt{\pi}}{2} = \frac{3\sqrt{\pi}}{4},$$

$$\Gamma\left(2+\frac{1}{2}+1\right) = \left(2+\frac{1}{2}\right)\Gamma\left(2+\frac{1}{2}\right) = \frac{5}{2}\frac{3\sqrt{\pi}}{4} = \frac{3\cdot 5\sqrt{\pi}}{2^3},$$

$$\Gamma\left(3+\frac{1}{2}+1\right) = \left(3+\frac{1}{2}\right)\Gamma\left(3+\frac{1}{2}\right) = \frac{3\cdot 5\cdot 7\sqrt{\pi}}{2^4},$$

and so on. In general, if n is a positive integer, then

$$\Gamma\left(n+\frac{1}{2}+1\right) = \frac{3\cdot 5\cdots(2n+1)\sqrt{\pi}}{2^{n+1}}.$$

Now back to the series for $J_{1/2}(x)$. We can now write

$$\begin{aligned}J_{1/2}(x) = \sqrt{\frac{x}{2}}\Bigg[&\frac{2}{\sqrt{\pi}} - \frac{2^2}{2^2 \cdot 3\sqrt{\pi}}x^2 \\ &+\frac{2^3}{2^4 2!3 \cdot 5\sqrt{\pi}}x^4 - \frac{2^4}{2^6 3!3 \cdot 5\dot{7}\sqrt{\pi}}x^6 \\ &+ \frac{2^5}{2^8 4! \cdot 3 \cdot 5 \cdot 7 \cdot 9\sqrt{\pi}}x^8 - \cdot\Bigg].\end{aligned}$$

This simplifies to

$$\begin{aligned}J_{1/2}(x) &= \sqrt{\frac{x}{2}}\left[\frac{2}{\sqrt{\pi}} - \frac{1}{3\sqrt{\pi}}x^2 + \frac{1}{2 \cdot 2 \cdot 3\dot{5}\sqrt{\pi}}x^4\right. \\ &\left.-\frac{1}{2^2 3!3 \cdot 5 \cdot 7\sqrt{\pi}}x^6 + \frac{1}{2^3 2 \cdot 3 \cdot 5 \cdot 7 \cdot 9\sqrt{\pi}}x^8 + \cdots\right] \\ &= \sqrt{\frac{x}{2}}\frac{1}{\sqrt{\pi}}\left[2 - \frac{1}{3}x^2 + \frac{1}{2 \cdot 2 \cdot 3 \cdot 5}x^4\right. \\ &\left.-\frac{1}{2^2 3! \cdot 3 \cdot 5 \cdot 7}x^6 + \frac{1}{2^3 \cdot 2 \cdot 3 \cdot 4 \cdot 3 \cdot 5 \cdot 7 \cdot 9}x^8 + \cdots\right].\end{aligned}$$

Finally, write this as

$$\begin{aligned}J_{1/2}(x) &= \sqrt{\frac{x}{2}}\frac{1}{\sqrt{\pi}}2\left[1 - \frac{1}{2 \cdot 3}x^2 + \frac{1}{2 \cdot 2 \cdot 2 \cdot 3 \cdot 5}x^4\right. \\ &\left.-\frac{1}{2^3 3!3 \cdot 5 \cdot 7}x^6 + \frac{1}{2^4 \cdot 2 \cdot 3 \cdot 4 \cdot 3 \cdot 5 \cdot 7 \cdot 9}x^8 \cdots\right] \\ &= \sqrt{\frac{2}{\pi x}}\left[x - \frac{1}{2 \cdot 3}x^3 + \frac{1}{2 \cdot 2 \cdot 2 \cdot 3 \cdot 5}x^5\right. \\ &\left.-\frac{1}{2^3 3! \cdot 3 \cdot 5 \cdot 7}x^7 + \frac{1}{2^4 \cdot 2 \cdot 3 \cdot 4 \cdot 3 \cdot 5 \cdot 7 \cdot 9}x^9 - \cdots\right] \\ &= \sqrt{\frac{2}{\pi x}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} \\ &= \sqrt{\frac{2}{\pi x}}\sin(x).\end{aligned}$$

A similar argument shows that

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos(x).$$

9. First, recall that $J_0'(x) = -J_1(x)$. Then

$$\int_0^{\alpha} J_1(\alpha s)\,ds == -J_0(x)\Big|_0^{\alpha} = J_0(0) - J_0(\alpha) = 1$$

because $J_0(0) = 1$ and $J_0(\alpha) = 0$ by choice of α. Now change variables by $s = \alpha x$ in the integral to get

$$\int_0^\alpha J_1(s)\, ds = \int_0^1 J_1(\alpha x)\alpha\, dx = 1$$

and this implies that

$$\int_0^1 J_1(\alpha x)\, dx = \frac{1}{\alpha}.$$

11. By equation (8.23),

$$(x^n J_n(x))' = x^n J_{n-1}(x).$$

Then

$$\int x^n J_{n-1}(x)\, dx = x^n J_n(x).$$

In similar fashion, equation (8.24) immediately yields the second integral.

13. Define

$$I_{n,k} = \int_0^1 (1 - x^2)^k x^{n+1} J_n(\alpha x)\, dx.$$

For part (a), begin with a result from Problem 11:

$$\int s^n J_{n-1}(s)\, ds = x^s J_n(s).$$

Replacing n with $n + 1$, we have

$$\int s^{n+1} J_n(s)\, ds = s^{n+1} J_{n+1}(s).$$

Then

$$\int_0^\alpha s^{n+1} J_n(s)\, ds = s^{n+1} J_{n+1}(s)\Big|_0^\alpha = \alpha^{n+1} J_{n+1}(\alpha).$$

Now let $s = \alpha x$ to get

$$\int_0^1 \alpha^{n+1} x^{n+1} J_n(\alpha x)\, dx = \alpha^{n+1} J_{n+1}(\alpha).$$

Then

$$\int_0^1 x^{n+1} J_n(\alpha x)\, dx = \frac{1}{\alpha} J_{n+1}(\alpha).$$

But,

$$I_{n,0} = \int_0^1 x^{n+1} J_n(\alpha x)\, dx.$$

Therefore

$$I_{n,0} = \frac{1}{\alpha} J_{n+1}(\alpha).$$

Now use the integral of Problem 12, with $n+1$ in place of n, to write

$$x^{n+1}J_n(\alpha x) = \frac{d}{dx}\left(\frac{1}{\alpha}x^{n+1}J_{n+1}(\alpha x)\right).$$

Upon substituting this into the definition of $I_{n,k}$, we have

$$I_{n,k} = \int_0^1 (1-x^2)^k \frac{d}{dx}\left(\frac{1}{\alpha}x^{n+1}J_{n+1}(\alpha x)\right)\,dx.$$

This completes part (b). For part (c), apply integration by parts to the integral of part (b):

$$\begin{aligned}
I_{n,k} &= \int_0^1 (1-x^2)^k \frac{d}{dx}\left(\frac{1}{\alpha}x^{n+1}J_{n+1}(\alpha x)\right)\,dx \\
&= (1-x^2)^k \frac{1}{\alpha}x^{n+1}J_{n+1}(\alpha x)\Big|_0^1 \\
&- \frac{1}{\alpha}\int_0^1 x^{n+1}J_{n+1}(\alpha x)k(1-x^2)^{k-1}(-2x)\,dx \\
&= \frac{2k}{\alpha}\int_0^1 (1-x^2)^{k-1}x^{n+2}J_{n+1}(\alpha x)\,dx \\
&= \frac{2k}{\alpha}I_{n+1,k-1}.
\end{aligned}$$

This relates $I_{n,k}$ to the value of this integral when n is increased by 1 and k is decreased by 1. In particular, if we carry out k repetitions of this operation, eventually increasing n to $n+k$, and decreasing k to k to 0, we obtain

$$\begin{aligned}
I_{n,k} &= \frac{2k}{\alpha}I_{n+1,k-1} \\
&= \frac{2k}{\alpha}\left[\frac{2(k-1)}{\alpha}I_{n+2,k-2}\right] \\
&= \frac{2^2k(k-1)}{\alpha^2}I_{n-2,k-2} \\
&= \frac{2^2k(k-1)}{\alpha^2}\left[\frac{2(k-2)}{\alpha}I_{n+3,k-2}\right] \\
&= \frac{2^3k(k-1)(k-2)}{\alpha^3}I_{n+3,k-3} \\
&= \cdots \frac{2^k k!}{\alpha^k}I_{n+k,0}.
\end{aligned}$$

Because k is a positive integer, we can write

$$k! = \Gamma(k+1)$$

in this expression.

For part (e), combine the results of parts (a) and (d) to write

$$\int_0^1 (1-x^2)^k x^{n+1} J_n(\alpha x)\,dx = \frac{2^k\Gamma(k+1)}{\alpha^{k+1}} J_{n+k+1}(\alpha).$$

For part (f), write this equation as

$$J_{n+k+1}(\alpha) = \frac{\alpha^{k+1}}{2^k\Gamma(k+1)} \int_0^1 (1-x^2)^k x^{n+1} J_n(\alpha x)\,dx.$$

The rest is just notation to provide the appropriate perspective. In the last equation, replace α with x and x with t to get

$$J_{n+k+1}(x) = \frac{x^{k+1}}{2^k\Gamma(k+1)} \int_0^1 t^{n+1}(1-t^2)^k J_n(xt)\,dt.$$

Finally, for part (g), let $m-n=k+1$ to get

$$J_m(x) = \frac{2x^{m-n}}{2^{m-n}\Gamma(m-n)} \int_0^1 t^{n+1}(1-t^2)^{m-n-1} J_n(xt)\,dt.$$

In these results, it is not necessary that k be an integer, because $k!$ has been replaced by $\Gamma(k+1)$, which is defined if $k+1>0$. For the expressions derived in this problem, it is enough to have $n+-1$, $k>-1$ and, in part (g),$m>n>-1$.

15. Start with the following result from Problem 14:

$$M_m(x) = \frac{x^m}{2^{m-1}\Gamma(m+1/2)} \int_0^1 (1-t^2)^{m-1/2}\cos(xt)\,dt.$$

Make the change of variables $t=\sin(\theta)$ to get

$$\begin{aligned} J_m(x) &= \frac{x^m}{2^{m-1}\Gamma(m+1/2)} \int_0^{\pi/2} (\cos^2(t))^{m-1/2}\cos(x\sin(\theta))\,d\theta \\ &= \frac{x^m}{2^{m-1}\Gamma(m+1/2)} \int_0^{\pi/2} \cos^{2m}(\theta)\cos(x\sin(\theta))\,d\theta. \end{aligned}$$

In Problems 17–24, the strategy is to match the given differential equation to the differential equation of Problem 16 by choosing a, b, c and ν. This makes it possible to write a general solution in terms of Bessel functions

17. The differential equation matches that of Problem 16 if

$$2a-1=-\frac{1}{3}, 2c-2=0, b^2c^2=1, \text{ and } a^2-\nu^2c^2=\frac{7}{144}.$$

Then

$$a=\frac{1}{3}, b=c=1, \text{ and } \nu=\frac{1}{4}.$$

We can write a general solution

$$y(x) = c_1x^{1/3}J_{1/4}(x) + c_2x^{1/3}J_{-1/4}(x).$$

19. Choose $a = 3, c = 4, b = 2, \nu = 1/2$ to get

$$y(x) = c_1 x^3 J_{1/2}(2x^4) + c_2 J_{-1/2}(2x^4).$$

21. Let $a = 2, c = 3, b = 1, \nu = 2/3$ to get

$$y(x) = c_1 x^2 J_{2/3}(x^3) + c_2 x^2 J_{-2/3}(x^3).$$

23. Here we get $a = b = 0$, so this method produces only the trivial solution. However, if the differential equation is multiplied by x^2, we obtain

$$x^2 y'' + xy' - \frac{1}{16} y = 0,$$

which is an Euler equation with general solution

$$y(x) = c_1 x^{1/4} + c_2 x^{-1/4}.$$

8.3 Some Applications of Bessel Functions

1. With $f(r) = r(1-r)$ and $g(r) = r^2$, the coefficients in the solution are

$$a_n = \frac{2}{J_1^2(j_n)} \int_0^1 Rs^2(1 - RS) J_0(j_n s)\, ds$$

and

$$b_n = \frac{R}{j_n c} \frac{2}{J_1^2(j_n)} \int_0^1 R^2 s^3 J_0(j_n s)\, ds.$$

The solution is

$$z(r,t) = \sum_{n=1}^{\infty} z_n(r,t),$$

where

$$z_n(r,t) = \left[a_n \cos\left(\frac{j_n ct}{R}\right) + b_n \sin\left(\frac{j_n ct}{R}\right)\right] J_0\left(\frac{j_n}{R} r\right).$$

Figures 8.11 through 8.14 show graphs of the first four normal modes of the solution times $t = 1/2, 1, 2$ and 4, for $R = 1$, $c = 2$, $f(r) = r(1-r)$ and $g(r) = 0$.

3. With $f(r) = r^2(1-r)$ and $g(r) = r + r^2$, the coefficients are

$$a_n = \frac{2}{J_1^2(j_n)} \int_0^1 R^2 s^3 (1 - Rs) J_0(j_n s)\, ds$$

and

$$b_n = \frac{R}{j_n c} \frac{2}{J_1^2(j_n)} \int_0^1 Rs^2(1 + Rs) J_0(j_n s)\, ds.$$

With $R = 1$, $c = 2$, $f(r) = r^2(1-r)$ and $g(r) = 0$, Figures 8.15–8.18 show the first four normal modes for times $t = 1/4, 1/2, 3/4$.

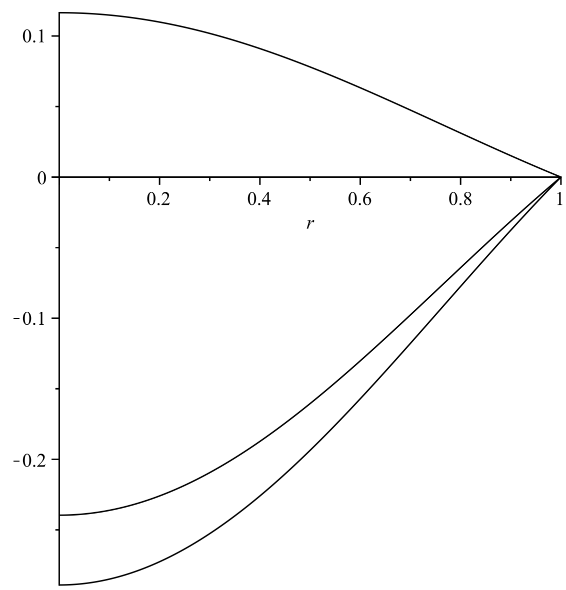

Figure 8.11: First normal mode in Problem 1 at times $t = 1/4, 1/2, 3/4$.

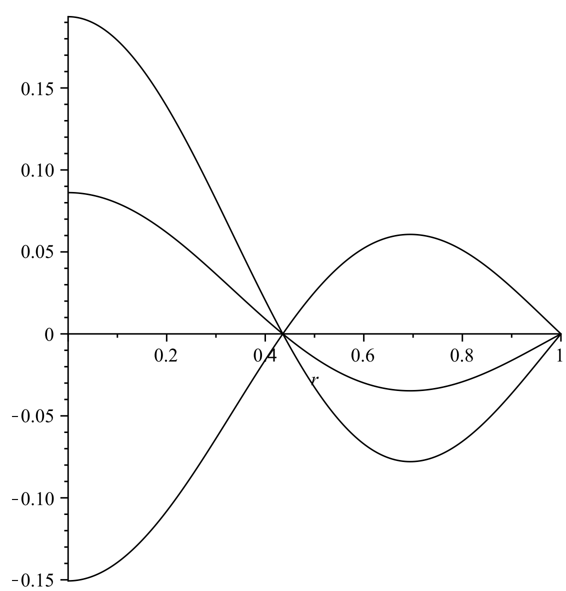

Figure 8.12: Second normal mode in Problem 1.

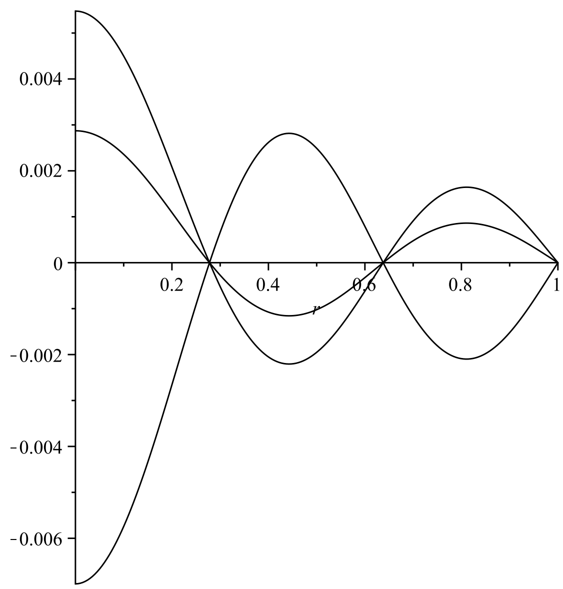

Figure 8.13: Third normal mode in Problem 1.

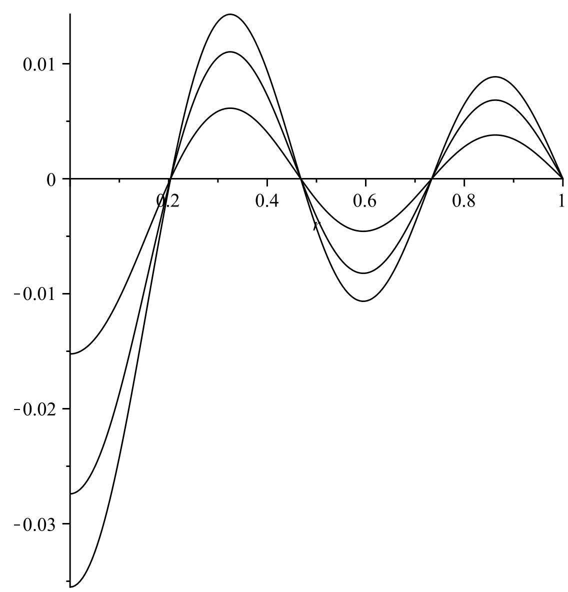

Figure 8.14: Fourth normal mode in Problem 1.

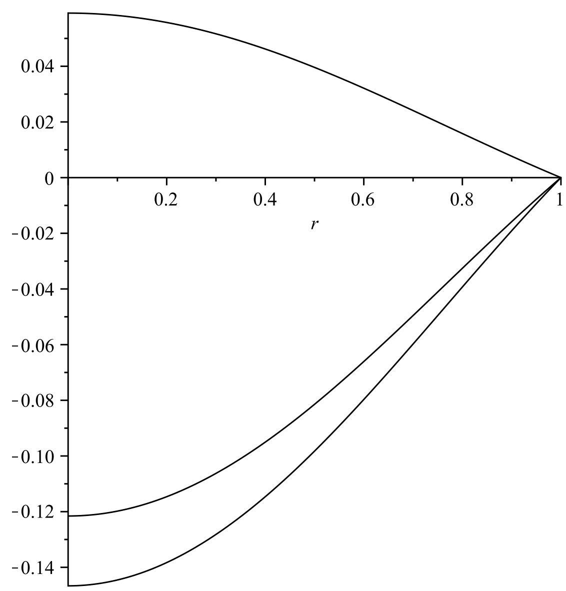

Figure 8.15: First normal mode in Problem 3.

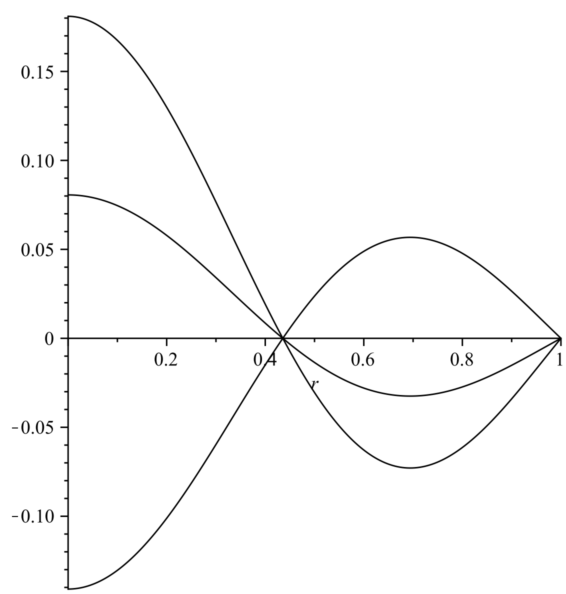

Figure 8.16: Second normal mode in Problem 3.

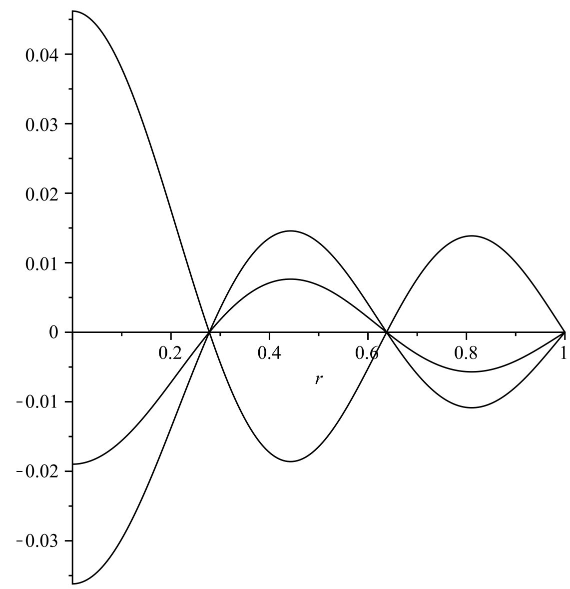

Figure 8.17: Third normal mode in Problem 3.

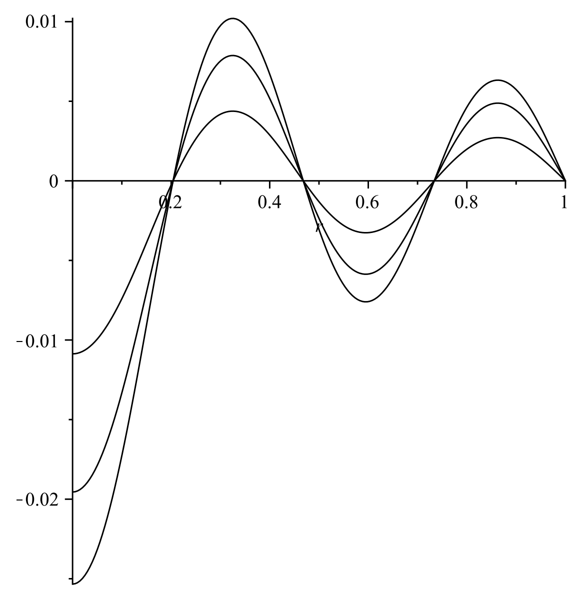

Figure 8.18: Fourth normal mode in Problem 3.

In each of Problems 5–8, the solution has the form

$$u(r,t) = \sum_{n=1}^{\infty} a_n J_0(j_n r) e^{-2j_n^2 t}.$$

For each, the expression for the coefficients is given.

5.
$$a_n = \frac{2}{J_1^2(j_n)} \int_0^1 \xi(1+\cos(\pi\xi)) J_0(j_n\xi)\, d\xi.$$

7.
$$a_n = \frac{2}{J_1^2(j_n)} \int_0^1 \xi^2 \cos(3\pi\xi/2) J_0(j_n\xi).$$

Chapter 9

Transform Methods of Solution

9.1 Laplace Transform Methods

1. Apply the Laplace transform with respect to t to the partial differential equation to get
$$s^2Y(x,s) = c^2Y''(x,s) + \frac{K}{s}.$$
Here primes denote differentiation with respect to x and the initial conditions have been inserted in using the operational rule for the derivatives. Write this equation as
$$Y'' - \frac{s^2}{c^2}Y = -\frac{K}{c^2 s}.$$
Think of this as a second-order differential equation in x, with s carried along as a parameter. The general solution is
$$Y(x,s) = c_1 e^{sx/c} + c_2 e^{-sx/c} + \frac{K}{s^3}.$$
Here c_1 and c_2 are "constants", but may involve s, because x is the variable of the differential equation. Now
$$Y(0,s) = [y(0,t)](s) = F(s) = c_1 + c_2 + \frac{K}{s^3}.$$
We need $\lim_{x\to\infty} y(0,t) = 0$, so $\lim_{s\to\infty} Y(x,s) = 0$. Therefore $c_1 = 0$ and
$$c_2 = F(s) - \frac{K}{s^3}.$$
We now have
$$Y(x,s) = \left(F(s) - \frac{K}{s^3}\right)e^{-sx/c} + \frac{K}{s^3}.$$

The solution is the inverse Laplace transform of $Y(x,s)$. Recalling the formula for the inverse Laplace transform of $e^{-as}F(s)$, we obtain

$$y(x,t) = \left[f\left(t-\frac{x}{c}\right) - \frac{K}{2}\left(t-\frac{x}{c}\right)^2\right] H\left(t-\frac{x}{c}\right) + \frac{1}{2}Kt^2,$$

in which H is the Heaviside function.

3. From the partial differential equation and the initial conditions,

$$s^2Y(x,s) = c^2Y'' - \frac{A}{s^2}.$$

Then

$$Y'' - \frac{s^2}{c^2} = \frac{A}{s^2},$$

with general solution

$$Y(x,s) = c_1e^{sx/c} + c_2e^{-sx/c} - \frac{A}{s^4}.$$

Because $\lim_{x\to\infty} y(x,t) = 0$, we must have $\lim_{s\to\infty} Y(x,s) = 0$, so $c_1 = 0$ and

$$Y(x,s) = c_2e^{-sx/c} - \frac{A}{s^4}.$$

Next, $y(0,t) = 0$, so

$$Y(0,s) = c_2 - \frac{A}{s^4}$$

and then

$$c_2 = \frac{A}{s^4}.$$

Then

$$Y(x,s) = \frac{A}{s^4}e^{-sx/c} - \frac{A}{s^4}.$$

The solution is the inverse of this,

$$y(x,t) = \frac{A}{6}\left(t-\frac{x}{c}\right)^3 H\left(t-\frac{x}{c}\right) - \frac{A}{6}t^3.$$

5. Transform the partial differential equation to get

$$s^2Y(x,s) = c^2Y''(x,s) - \frac{Ax}{s^2}.$$

Then

$$Y'' - \frac{s^2}{c^2}Y = \frac{Ax}{c^2s^2}.$$

This has general solution

$$Y(x,s) = c_1e^{sx/c} + c_2e^{-sx/c} - \frac{Ax}{s^4}.$$

The condition that $\lim_{x\to\infty} y(x,t) = 0$ forces $\lim_{s\to\infty} Y(x,s) = 0$, so $c_1 = 0$ and

$$Y(x,t) = c_2 e^{-sx/c} - \frac{Ax}{s^4}.$$

Next,

$$\mathcal{L}[y(0,t)](s) = F(s) = Y(0,s) = c_2.$$

Then

$$Y(x,s) = F(s)e^{-sx/c} - \frac{Ax}{s^4}.$$

Invert this for the solution

$$y(x,t) = f\left(t - \frac{x}{c}\right) H\left(t - \frac{x}{c}\right) - \frac{1}{6}Axt^3.$$

7. Take the transform with respect to t of the heat equation to get

$$sU(x,s) - e^{-x} = kU''(x,s),$$

or

$$U'' - \frac{s}{k}U = -\frac{1}{k}e^{-x}.$$

The associated homogeneous equation of this nonhomogeneous equation has the general solution

$$U_h(x,s) = c_1 e^{\sqrt{s/k}x} + c_2 e^{-\sqrt{s/k}x}.$$

For a particular solution, use undetermined coefficients, trying

$$U_p(x,s) = Ae^{-x}.$$

Substitute this into the nonhomogeneous differential equation to get

$$A - \frac{s}{k}A = -\frac{1}{k},$$

so

$$A = \frac{s-k}{.}$$

Then,

$$U(x,s) = U_h(x,s) + U_p(x,s) = c_1 e^{\sqrt{s/k}x} + c_2 e^{-\sqrt{s/k}x} + \frac{1}{s-k}e^{-x}.$$

Because $\lim_{x\to\infty} u(x,t) = 0$, choose $c_1 = 0$, so

$$U(x,s) = c_2 e^{-\sqrt{s/k}x} + \frac{1}{s-k}e^{-x}.$$

Take the transform of $u(0,t) = 0$ to get

$$U(0,s) = c_2 + \frac{1}{s-k}.$$

Then
$$c_2 = -\frac{1}{s-k}$$
so
$$U(x,s) = -\frac{1}{s-k}e^{-\sqrt{s/k}x} + \frac{1}{s-k}e^{-x}.$$
Using the convolution theorem, write
$$u(x,t) = -e^{-kt} * \mathcal{L}^{-1}\left[e^{-\sqrt{s/k}x}\right](t) + e^{kt}e^{-x}.$$
By consulting a table, we obtain
$$\mathcal{L}^{-1}\left[e^{-(x/\sqrt{k})s}\right](t) = \frac{x}{2\sqrt{\pi k t^3}}e^{-x^2/4kt}.$$
Then,
$$u(x,t) = -e^{-kt} * \frac{x}{2\sqrt{\pi k t^3}}e^{-x^2/4kt} + e^{kt-x}.$$

9.2 Fourier Transform Methods

For the first four problems, the solution is given by
$$u(x,t) = \frac{1}{2\sqrt{\pi k t}}\int_{-\infty}^{\infty} f(\xi)e^{-(x-\xi)^2/4kt}.$$

1. With $f(x) = e^{-4|x|}$, the solution is
$$\frac{1}{2\sqrt{\pi k t}}\int_{-\infty}^{\infty} e^{-4|\xi|}e^{-(x-\xi)^2/4kt}\,d\xi.$$

3.
$$u(x,t) = \frac{1}{2\sqrt{\pi k t}}\int_{0}^{4} \xi e^{-(x-\xi)^2/4kt}\,d\xi.$$

5. Take the Fourier transform of the wave equation with respect to x to get
$$\widehat{y}'' = 144(i\omega)^2\widehat{y} = -144\omega^2\widehat{y},$$
or
$$\widehat{y}'' + 144\omega^2\widehat{y} = 0.$$
Then
$$\widehat{y}(\omega,t) = c_1\cos(12\omega t) + c_2\sin(12\omega t),$$
in which primes denote differentiation with respect to t. Because $y_t(x,0) = 0$, then $c_1 = 0$ and
$$\widehat{y}(\omega,t) = c_1\cos(12\omega t).$$

Next, $y(x,0) = f(x)$, so

$$\widehat{y}(\omega,0) = \widehat{f}(\omega).$$

Then $c_1 = \widehat{f}(\omega)$, so

$$\widehat{y}(\omega,t) = \widehat{f}(\omega)\cos(12\omega t).$$

It is routine to compute (or use a software routine to find)

$$\widehat{f}(\omega) = \frac{10}{25+\omega^2}.$$

Then

$$\widehat{y}(\omega,t) = \frac{10}{25+\omega^2}\cos(144\omega t).$$

Finally, use the integral formula for the inverse Fourier transform to obtain

$$y(x,t) = \operatorname{Re}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{10}{25+\omega^2}\cos(12\omega t)e^{i\omega x}\,d\omega\right].$$

Of course $y(x,t)$ is a real quantity, so in the last line we have taken the real part of the integral. If we replace

$$e^{i\omega x} = \cos(\omega x) + i\sin(\omega x)$$

in this integral, we can write more explicitly

$$y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{10}{25+\omega^2}\cos(\omega x)\cos(12\omega t)\,d\omega.$$

7. Apply the Fourier transform to the initial-boundary value problem to get

$$\begin{aligned}
&\widehat{y}'' + 16\omega^2\widehat{y} = 0,\\
&\widehat{y}(\omega,0) = 0,\\
&\widehat{y}'(\omega,0) = \int_{-\pi}^{\pi}\sin(\xi)e^{-i\omega\xi}\,d\xi = \frac{2i\sin(\pi\omega)}{\omega^2-1}.
\end{aligned}$$

The solution of the transformed problem is

$$\widehat{y}(\omega,t) = \frac{2i\sin(\pi\omega)}{4\omega(\omega^2-1)}\sin(4\omega t).$$

Invert this to obtain the solution

$$y(x,t) = \operatorname{Re}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{i\sin(\pi\omega)}{2\omega(\omega^2-1)}\sin(4\omega t)e^{i\omega x}\,d\omega\right].$$

9. The problem transforms (in x) to

$$\begin{aligned}&\widehat{y}'' + 9\omega^2\widehat{y} = 0,\\&\widehat{y}(\omega, 0) = 0,\\&\widehat{y}'(\omega, 0) = \mathcal{F}[e^{-2x}H(x-1)](\omega) = \frac{(2-i\omega)e^{-(2+i\omega)}}{4+\omega^2}.\end{aligned}$$

This problem has the solution

$$\widehat{y}(\omega, t) = \frac{(2-i\omega)e^{-(2+i\omega)}}{3\omega(4+\omega^2)}.$$

Then

$$y(x,t) = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} \frac{(2-i\omega)e^{-(2+i\omega)}}{3\omega(4+\omega^2)} \sin(3\omega t)e^{i\omega x}\, d\omega\right].$$

11. This problem was solved by the Fourier transform in the text, so we can use the result to immediately write

$$\begin{aligned}u(x,y) &= \frac{y}{\pi}\left[\int_{-4}^{0} \frac{-1}{y^2+(\xi-x)^2}\, d\xi + \int_0^4 \frac{1}{y^2+(\xi-x)^2}\, d\xi\right]\\&= -\frac{y}{\pi}\left[\arctan\left(\frac{x+4}{y}\right) + \arctan\left(\frac{x-4}{y}\right)\right].\end{aligned}$$

9.3 Fourier Sine and Cosine Transform Methods

1. Take a Fourier sine transform (in x) of the problem to get

$$\begin{aligned}&\widehat{y}_S'' + 9\omega^2\widehat{y}_S = 0,\\&\widehat{y}_S(\omega, 0) = \int_0^1 \xi(1-\xi)\sin(\omega\xi)\, d\xi = \frac{2(1-\cos(\omega) - \omega\sin(\omega)}{\omega^3},\\&\widehat{y}_S'(\omega, 0) = 0.\end{aligned}$$

The solution of the transformed problem is

$$\widehat{y}_S(\omega, t) = \left[\frac{2(1-\cos(\omega)) - \omega\sin(\omega)}{\omega^3}\right]\cos(3\omega t).$$

Invert this to obtain the solution

$$y(x,t) = \frac{2}{\pi}\int_0^{\infty}\left[\frac{2(1-\cos(\omega)) - \omega\sin(\omega)}{\omega^3}\right]\sin(\omega x)\cos(3\omega t)\, d\omega.$$

3. The sine transformed problem is

$$\begin{aligned}&\widehat{y}_S'' + 4\omega^2\widehat{y}_S = 0,\\&\widehat{y}_S(\omega, 0) = 0,\\&\widehat{y}_S'(\omega, 0) = \int_{\pi/2}^{5\pi/2} \cos(\xi)\sin(\omega\xi)\, d\xi = \frac{\sin(\omega\pi/2) - \sin(5\omega\pi/2)}{\omega^2 - 1}.\end{aligned}$$

This has solution

$$\widehat{y}_S(\omega,t) = \frac{\sin(\omega\pi/2) - \sin(5\omega\pi/2)}{2\omega(\omega^2-1)}\sin(2\omega t).$$

Invert this to obtain the solution

$$y(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\sin(\omega\pi/2) - \sin(5\omega\pi/2)}{2\omega(\omega^2-1)}\sin(\omega x)\sin(2\omega t).$$

5. After applying the sine transform to the initial-boundary value problem, we obtain

$$\begin{aligned}
&\widehat{y}_S'' + 196\omega^2\widehat{y}_S = 0,\\
&\widehat{y}_S(\omega,0) = 0,\\
&\widehat{y}_S'(\omega,0) = \int_0^3 \xi^2(3-\xi)\sin(\omega\xi)\,d\xi\\
&= \frac{3}{\omega^4}(2\sin(3\omega) - 4\omega\cos(3\omega) - 3\omega^2\sin(3\omega) - 2\omega).
\end{aligned}$$

The transformed problem has the solution

$$\begin{aligned}
&\widehat{y}_S(\omega,t) =\\
&\frac{3}{14\omega^5}(2\sin(3\omega) - 4\omega\cos(3\omega) - 3\omega^2\sin(3\omega) - 2\omega)\sin(14\omega t).
\end{aligned}$$

Then

$$y(x,t) = \frac{2}{\pi}\int_0^\infty \widehat{y}_S(\omega,t)\sin(\omega x)\,d\omega.$$

7. Apply the Fourier cosine transform in x to the problem to get

$$\widehat{u}_C' + (1+\omega^2)\widehat{u}_C = -f(t); \widehat{u}_C(\omega,0) = 0.$$

This has the solution

$$\begin{aligned}
\widehat{u}_C(\omega,t) &= e^{-(1+\omega^2)t}\int_0^t f(\tau)e^{(1+\omega^2)\tau}\,d\tau\\
&= -f(\tau) * e^{-(1+\omega^2)t}.
\end{aligned}$$

Invert this to obtain

$$u(x,t) = -\frac{2}{\pi}\int_0^\infty f(t) * e^{-(1+\omega^2)t}\cos(\omega x)\,d\omega.$$

Chapter 10

Vectors and the Vector Space R^n

10.1 Vectors in the Plane and $3-$Space

1.
$$\mathbf{F}+\mathbf{G}=(2+\sqrt{2})\mathbf{i}+3\mathbf{j}, \mathbf{F}-\mathbf{G}=(2-\sqrt{2})\mathbf{i}-9\mathbf{j}+10\mathbf{k},$$
$$2\mathbf{F}=4\mathbf{i}-6\mathbf{j}+10\mathbf{k}, 3\mathbf{G}=3\sqrt{2}\mathbf{i}+18\mathbf{j}-15\mathbf{k}, \| \mathbf{F} \|=\sqrt{38}$$

3.
$$\mathbf{F}+\mathbf{G}=3\mathbf{i}-\mathbf{k}, \mathbf{F}-\mathbf{G}=\mathbf{i}-10\mathbf{j}+\mathbf{k},$$
$$2\mathbf{F}=2\mathbf{i}-6\mathbf{k}, 3\mathbf{G}=3\mathbf{i}+15\mathbf{j}-3\mathbf{k}, \| \mathbf{F} \|=\sqrt{29}$$

5.
$$\mathbf{F}+\mathbf{G}=3\mathbf{i}-\mathbf{j}+3\mathbf{k}, \mathbf{F}-\mathbf{G}=-\mathbf{i}+3\mathbf{j}-\mathbf{k},$$
$$2\mathbf{F}=2\mathbf{i}+2\mathbf{j}+2\mathbf{k}, 3\mathbf{G}=6\mathbf{i}-6\mathbf{j}+6\mathbf{k}, \| \mathbf{F} \|=\sqrt{3}$$

7.
$$\frac{9}{\sqrt{45}}(-5\mathbf{i}-4\mathbf{j}+2\mathbf{k})$$

9.
$$\frac{4}{9}(-4\mathbf{i}+7\mathbf{j}+4\mathbf{k})$$

11. $x=3-6t, y=1-t, z=0$

13. $x=0, y=1-t, z=3-2t$

15. $x=2-3t, y=-3+9t, z=6-2t$

10.2 The Dot Product

In Problems 1–6, **F** is the first vector given in the problem **G** is the second, and θ is the angle between **F** and **G**.

1. $\mathbf{F} \cdot \mathbf{G} = 2$ and
$$\cos(\theta) = \frac{\mathbf{F} \cdot \mathbf{G}}{\| \mathbf{F} \| \| \mathbf{G} \|} = \frac{2}{\sqrt{14}}.$$
F and **G** are not orthogonal.

3. $\mathbf{F} \cdot \mathbf{G} = -23, \cos(\theta) = -23/\sqrt{29}\sqrt{41}$ and the vectors are not orthogonal.

5. $\mathbf{F} \cdot \mathbf{G} = -18, \cos(\theta) = -9/10$ and the vectors are not orthogonal.

In Problems 7–12, if the given point is (x_0, y_0, z_0) and the normal to the proposed plane is $\mathbf{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then the plane through the point and having **N** as normal vector has equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

The constant terms are usually collected to write this equation in the form

$$ax + by + cz = k.$$

7. The plane has equation
$$3(x + 1) - (y - 1) + 4(z - 2) = 0,$$
or
$$3x - y + 4z = 4.$$

9. $4x - 3y + 2z = 25$

11. $7x + 6y - 5z = -26$

In each of Problems 13–17, the projection of **v** onto **u** is the vector

$$\frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \|^2}\mathbf{u}.$$

13.
$$\text{proj}_{\mathbf{u}}\mathbf{v} = -\frac{9}{14}\mathbf{u}$$

15.
$$\frac{1}{62}\mathbf{u}$$

17.
$$\frac{15}{53}\mathbf{u}$$

10.3 The Cross Product

1.
$$\mathbf{F}\times\mathbf{G}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -3 & 6 & 1\\ -1 & -2 & 1\end{vmatrix}=8\mathbf{i}+2\mathbf{j}+12\mathbf{k}$$

and

$$\mathbf{G}\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & -2 & 1\\ -3 & 6 & 1\end{vmatrix}=-8\mathbf{i}-2\mathbf{j}-12\mathbf{k}=-\mathbf{F}\times\mathbf{G}$$

3.
$$\mathbf{F}\times\mathbf{G}=-8\mathbf{i}-12\mathbf{j}-5\mathbf{k}=-\mathbf{G}\times\mathbf{F}$$

In Problems 5–9, the three given points are used to find two nonparallel vectors in the plane that is sought (assuming that the points are not collinear). The cross product of these vectors is a normal to the plane. We than have a point on the plane and a normal to the plane, so we can find an equation of the plane.

5. Vectors from the first point to the second and third points are $\mathbf{F}=4\mathbf{i}-\mathbf{j}-6\mathbf{k}$ and $\mathbf{G}=\mathbf{i}-\mathbf{k}$. Compute

$$\mathbf{N}=\mathbf{F}\times\mathbf{G}=\mathbf{i}-2\mathbf{j}+\mathbf{k}.$$

Because this cross product is not the zero vector, the given points are not collinear. The plane containing these points has equation

$$((x+1)\mathbf{i}+(y-1)\mathbf{j}+(z-6)\mathbf{k})\cdot\mathbf{N}=0.$$

This is

$$x+1-2(y-1)+z-6=0,$$

or

$$x-2y+z=3.$$

7. $2x-11y+z=0$

9. $29x+37y-12z=30$

In Problems 10–12, recall that a plane $ax+by+cz=k$ has normal vector $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$, or any nonzero multiple of this vector.

11. $\mathbf{N}=\mathbf{i}-\mathbf{j}+2\mathbf{k}$

13. If two sides of a parallelogram meet at a point, with an angle θ between the sides, then the area of the parallelogram is the product of the lengths of these sides, times the cosine of θ. Now suppose $\mathbf{F}$ and $\mathbf{G}$ are vectors

drawn from a corner of the parallelogram, along two incident sides, and θ the angle between these vectors. Then

$$\| \mathbf{F} \|\| \mathbf{G} \| \cos(\theta)$$

is the product of the lengths of incident sides, times the angle between them, hence is the area of the parallelogram. But

$$\| \mathbf{F} \|\| \mathbf{G} \| \cos(\theta) = \| \mathbf{F} \times \mathbf{G} \|$$

so the magnitude of this cross product is the area of the parallelogram.

10.4 $n-$ Vectors and the Algebraic Structure of R^n

1. Let $\mathbf{F} =< -2, 1, -1, 4 >$. Then $\mathbf{O} =< 0, 0, 0, 0 >$ is in S (the scalar multiple of 0 with $\mathbf{F}$). Further, if a and b are real numbers, then

$$a\mathbf{F} + b\mathbf{F} = (a + b)\mathbf{F},$$

so a sum of scalar multiples of $\mathbf{F}$ is again a scalar multiple of $\mathbf{F}$, hence is in S. Finally,

$$a(b\mathbf{F}) = (ab)\mathbf{F}$$

so a scalar multiple of a vector in S is in S. Therefore S is a subspace of R^4.

3. S is not a subspace of R^5. The zero vector does not have fourth component 1, and a sum of vectors with fourth component 1 does not have fourth component 1. S fails on scalar multiples as well.

5. S is not a subspace of R^4. For example, $< 1, 1, 1, 0 >$ and $< 0, 1, 1, 1 >$ are both in S, but their sum is not, having no zero component.

In Problems 7–16, keep in mind that a set of vectors is linearly dependent if some linear combination of these vectors, with at least one nonzero coefficient, is equal to the zero vector. And the vectors are linearly independent exactly when the only linear combination of these vectors adding up to the zero vector is the trivial one, with all zero coefficients.

7. If

$$a(3\mathbf{i} + 2\mathbf{j}) + b(\mathbf{i} - \mathbf{j}) =< 0, 0 >,$$

then

$$3a + b = 0 \text{ and } 2a - b = 0.$$

From the second equation, $b = 2a$, and then the first equation is $5a = 0$, so $a = 0$ and then $b = 0$. The only linear combination of these vectors that equals the zero vector is the trivial combination, so the vectors are linearly independent.

We could also observe that neither vector is a linear combination of the other, which would be the case of these two vectors were linearly dependent.

9. These two vectors are linearly independent because neither is a scalar multiple of the other (which would occur for two linearly dependent vectors).

11. The vectors are linearly dependent because
$$2 < 1, 2, -3, 1 > + < 4, 0, 0, 2 > - < 6, 4, -6, 4 >=< 0, 0, 0, 0 > .$$

13. The vectors are linearly dependent because
$$2 < 1, -2 > -2 < 4, 1 > + < 6, 6 >=< 0, 0 > .$$

15. The vectors are linearly independent.

In each of Problems 17–21 it is routine to show that S is not empty, and that a linear combination of vectors of S is in S, showing that S is a subspace of the appropriate R^n. We will show how to find a basis for S.

17. Every vector in S has the form
$$x < 1, 0, 0, -1 > +y < 0, 1, -1, 0 > .$$
Therefore the two vectors $< 1, 0, 0, -1 >$ and $< 0, 1, -1, 0 >$ span S. These vectors are also linearly independent, and so form a basis for S, which has dimension 2.

19. S consists of all vectors in R^n of the form
$$< x_1, 0, x_2, \cdots, x_{n-1} > .$$
The $n-1$ independent vectors
$$< 1, 0, 0, \cdots, 0 >, < 0, 0, 1, 0, \cdots, 0 >, \cdots, < 0, 0, 0, \cdots, 0, 1 >$$
span S, so S has dimension $n-1$.

21. Every vector in S is a scalar multiple of
$$0, 1, 0, 2, 0, 3, 0 >$$
so this vector forms a basis for S and S has dimension 1.

23. The spanning vectors are independent and form a basis for the subspace S of R^3 that they span. Further, by inspection,
$$< -5, -3, -3 >= -5 < 1, 1, 1 > +2 < 0, 1, 1 > .$$

25. The spanning vectors are independent and so form a basis for the subspace S of R^4 that they span. This subspace has dimension 3. For $\mathbf{X}$ to be in S, we need numbers a, b and c so that

$$a < 1, 0, -3, 2 > +b < 1, 0, -1, 1 >=< -4, 0, 10, -5 > .$$

This requires that

$$a + b = -4, -3a + b = 10, \text{ and } 2a + b = -5.$$

Then $a = -3, b = -1$, so

$$< -4, 0, 10, -7 >= -3 < 1, 0, -3, 2 > - < 1, 0, -1, 1 >$$

and $< -4, 0, 10, -4 >$ is in S.

27. Because $\mathbf{U}$ is in S, which is spanned by $\mathbf{V}_1, \cdots, \mathbf{V}_k$, we must have

$$\mathbf{U} = c_1\mathbf{V}_1 + \cdots + c_k\mathbf{V}_k,$$

so $\mathbf{V}_1, \cdots, \mathbf{V}_k, \mathbf{U}$ are linearly independent.

29. Suppose the set of vectors is $\mathbf{V}_1, \cdots, \mathbf{V}_k, \mathbf{O}$. Then

$$0\mathbf{V}_1 + 0\mathbf{V}_2 + \cdots + 0\mathbf{V}_k + 1\mathbf{O} = \mathbf{O}$$

is a linear combination of the vectors adding up to the zero vector, and with a nonzero coefficient. This makes the original set of vectors (including the zero vector) linearly dependent.

10.5 Orthogonal Sets and Orthogonalization

1.

$$\begin{aligned}
&\| \mathbf{V}_1 + \cdots + \mathbf{V}_k \|^2 = (\mathbf{V}_1 + \cdots + \mathbf{V}_k) \cdot (\mathbf{V}_1 + \cdots + \mathbf{V}_k) \\
&= \mathbf{V}_1 \cdot (\mathbf{V}_1 + \cdots + \mathbf{V}_k) + \mathbf{V}_2 \cdot (\mathbf{V}_1 + \cdots + \mathbf{V}_k) + \cdots \\
&+ \mathbf{V}_k \cdot (\mathbf{V}_1 + \cdots + \mathbf{V}_k) \\
&= \mathbf{V}_1 \cdot \mathbf{V}_1 + \mathbf{V}_2 \cdot \mathbf{V}_2 + \cdots + \mathbf{V}_k \cdot \mathbf{V}_k \\
&= \| \mathbf{V}_1 \|^2 + \| \mathbf{V}_2 \|^2 + \cdots \| \mathbf{V}_k \|^2,
\end{aligned}$$

in which we have used the fact that $\mathbf{V}i \cdot \mathbf{V}_j = 0$ if $i \neq j$.

3. Reason as in Problem 2, except now use the fact that

$$\sum_{j=1}^{n} (\mathbf{X} \cdot \mathbf{V}_j)\mathbf{V}_j = \mathbf{X}.$$

In each of Problem 4–11, the given vectors are denoted $\mathbf{X}_1, \cdots, \mathbf{X}_k$ in the given order.

5. Take $\mathbf{V}_1 = \mathbf{X}_1$ and
$$\mathbf{V}_2 = \mathbf{X}_2 + \frac{11}{5}\mathbf{X}_1 =< 0, 4/5, 2/5, 0 > .$$

7. Let $\mathbf{V}_1 = \mathbf{X}_1$ and
$$\begin{aligned}\mathbf{V}_2 &= \mathbf{X}_2 + \frac{5}{26}\mathbf{X}_1 \\ &= \frac{1}{26} < 109, 0, -41, 58 > .\end{aligned}$$

   Finally, let
$$\begin{aligned}\mathbf{V}_3 &= \mathbf{X}_3 - \frac{17}{26}\mathbf{X}_1 - \frac{331/26}{651/26}\mathbf{V}_1 \\ &= \mathbf{X}_3 - \frac{17}{26}\mathbf{V}_1 - \frac{331}{651}\mathbf{V}_2 \\ &= \frac{1}{651} < -962, 0, -1406, 0, 814 > .\end{aligned}$$

9. $\mathbf{V}_1 = \mathbf{X}_1$,
$$\begin{aligned}\mathbf{V}_2 &= \mathbf{X}_2 - \frac{1}{10}\mathbf{X}_1 \\ &= \frac{1}{10} < 21, -8, -60, -31, -18, 0 >,\end{aligned}$$

$$\begin{aligned}\mathbf{V}_3 &= \mathbf{X}_3 - \frac{3}{10}\mathbf{X}_1 - \frac{163/10}{269/10}\mathbf{V}_2 \\ &= \frac{1}{269} < -423, -300, 489, -759, 132, 0 >,\end{aligned}$$

   and
$$\begin{aligned}\mathbf{V}_4 &= \mathbf{X}_4 + \frac{15}{10}\mathbf{X}_1 - \frac{13/2}{260/10}\mathbf{V}_2 - \frac{4455/269}{4095/269}\mathbf{V}_4 \\ &= \frac{1}{91} < 337, -145, 250, 29, -9, 0 > .\end{aligned}$$

11. $\mathbf{V}_1 = \mathbf{X}_1$ and $\mathbf{V}_2 = \mathbf{X}_2$ because $\mathbf{X}_2$ and $\mathbf{X}_1$ are orthogonal. Finally,
$$\mathbf{V}_3 = \mathbf{X}_3 + \frac{4}{12}\mathbf{V}_1 + \frac{4}{2}\mathbf{V}_2 = \frac{1}{3} < 0, -8, 0, -8, 0, 16 > .$$

10.6 Orthogonal Complements and Projections

1. Let $\mathbf{V}_1 =< 1, -1, 0, 0 >$ and $\mathbf{V}_2 =< 1, 1, 0, 0 >$. These form an orthogonal basis for S. Then

$$\begin{aligned}\mathbf{u}_S &= \frac{\mathbf{u}\cdot\mathbf{V}_1}{\mathbf{V}_1\cdot\mathbf{V}_1}\mathbf{V}_1 + \frac{\mathbf{u}\cdot\mathbf{V}_2}{\mathbf{V}_2\cdot\mathbf{V}_2}\mathbf{V}_2 \\ &= -4\mathbf{V}_1 + 2\mathbf{V}_2 =< -2, 6, 0, 0 >\end{aligned}$$

and

$$\mathbf{u}^{\perp} = \mathbf{u} - \mathbf{u}_S =< 0, 0, 1, 7 > .$$

The distance between $\mathbf{u}$ and S is

$$\| \mathbf{u}^{\perp} \| = \sqrt{50}.$$

3. Let

$$\mathbf{V}_1 =< 1, -1, 0, 1, -1 >, \mathbf{V}_2 =< 1, 0, 0, -1, 0 >, \mathbf{V}_3 =< 0, -1, 0, 0, 1 > .$$

Then

$$\mathbf{u}_S = \frac{7}{2}\mathbf{V}_1 + \mathbf{V}_2 - 3\mathbf{V}_3 = \frac{1}{2} < 9, -1, 0, 5, -13 > .$$

and

$$\mathbf{u}^{\perp} = \frac{1}{2} < -1, -1, 6, -1, -1 > .$$

The distance between $\mathbf{u}$ and S is

$$\| \mathbf{u}^{\perp} \| = \sqrt{10}.$$

5. Let

$$\mathbf{V}_1 =< 1, 0, 1, 0, 1, 0, 0 > \text{ and } \mathbf{V}_2 =< 0, 1, 0, 1, 0, 0, 0 > .$$

We find that

$$\mathbf{u}_S = 3\mathbf{V}_1 + \frac{1}{2}\mathbf{V}_2 = \frac{1}{2} < 6, 1, 6, 1, 6, 0, 0 >$$

and

$$\mathbf{u}^{\perp} = \frac{1}{2} < 10, 1, -4, -1, -6, -6, 8 > .$$

The distance between $\mathbf{u}$ and S is

$$\| \mathbf{u}^{\perp} \| = \frac{1}{2}\sqrt{254}.$$

7. Let $\mathbf{v}_1, \cdots, \mathbf{v}_k$ be an orthogonal basis for S, and $\mathbf{u}_1, \cdots \mathbf{u}_r$ an orthogonal basis for $S^{\perp}$. Then the vectors

$$\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{v}_1, \cdots \mathbf{v}_r$$

span R^n because every $n-$ vector is a sum of a vector in S (a linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_k$) and a vector in $S^\perp$ (a linear combination of $\mathbf{u}_1, \cdots, \mathbf{u}_r$). Further, the set of vectors

$$\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{v}_1, \cdots \mathbf{v}_r$$

are linearly independent, because no one of them is a linear combination of others in this set. These $k + r$ vectors therefore form a basis for R^n. Because R^n has dimension n,

$$n = k + r.$$

That is,

$$\text{dimension}(S) + \text{ dimension } (S^\perp) = \text{ dimension}(R^n).$$

9. Denote

$$\mathbf{V}_1 =< 2, 1, -1, 0, 0 >, \mathbf{V}_2 =< -1, 2, 0, 1, 0 >, \mathbf{V}_3 =< 0, 1, 1, -2, 0 > .$$

These form an orthogonal basis for S. With $\mathbf{u} =< 4, 3, -3, 4, 7 >$, compute

$$\begin{aligned} \mathbf{u}_S &= \frac{7}{3}\mathbf{V}_1 + \mathbf{V}_2 - \frac{4}{3}\mathbf{V}_3 \\ &= \frac{1}{3} < 11, 9, -11, 11, 0 > . \end{aligned}$$

10. Denote

$$\mathbf{V}_1 =< 0, 1, 1, 0, 0, 1 >, \mathbf{V}_2 =< 0, 0, 3, 0, 0, -3 >, \mathbf{V}_3 =< 6, 0, 0, -2, 0, 0 > .$$

With $\mathbf{u} =< 0, 1, 1, -2, -2, 6 >$, compute

$$\begin{aligned} \mathbf{u}_S &= \frac{8}{3}\mathbf{V}_1 - \frac{5}{6}\mathbf{V}_2 + \frac{1}{10}\mathbf{V}_3 \\ &=< 3/5, 8/3, 1/6, -1/5, 0, 31/6 > . \end{aligned}$$

This is the vector in S closest to $\mathbf{u}$.

Chapter 11

Matrices, Determinants and Linear Systems

11.1 Matrices and Matrix Algebra

1.

$$2\mathbf{A} - 3\mathbf{B} = \begin{pmatrix} 14 & -2 & 6 \\ 10 & -5 & -6 \\ -26 & -43 & -8 \end{pmatrix}$$

3.

$$\mathbf{A}^2 + 2\mathbf{AB} = \begin{pmatrix} 2 + 2x - x^2 & 12x + (1-x)(x + e^x + 2\cos(x)) \\ 4 + 2x + 2e^x + 2xe^x & -22 - 2x + e^{2x} + 2e^x\cos(x) \end{pmatrix}$$

5.

$$4\mathbf{A} + 8\mathbf{B} = \begin{pmatrix} -36 & 0 & 68 & 196 & 20 \\ 128 & -40 & -36 & -8 & 72 \end{pmatrix}$$

7. $\mathbf{BA}$ is not defined;

$$\mathbf{AB} = \begin{pmatrix} -10 & -34 & -16 & -30 & -14 \\ 10 & -2 & -11 & -8 & -45 \\ -5 & 1 & 15 & 61 & -63 \end{pmatrix}$$

9. $\mathbf{AB} = (115)$;

$$\mathbf{BA} = \begin{pmatrix} 3 & -18 & -6 & -42 & 66 \\ -2 & 12 & 4 & 28 & -44 \\ -6 & 36 & 12 & 84 & -132 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & -24 & -8 & -56 & 88 \end{pmatrix}$$

11. **AB** is not defined;

$$\mathbf{BA} = \begin{pmatrix} 410 & 36 & -56 & 227 \\ 17 & 253 & 40 & -1 \end{pmatrix}$$

13. **AB** is not defined;

$$\mathbf{BA} = \begin{pmatrix} -16 & -13 & -5 \end{pmatrix}$$

15. **BA** is not defined;

$$\mathbf{AB} = \begin{pmatrix} 39 & -84 & 21 \\ -23 & 38 & 3 \end{pmatrix}$$

17. **AB** is 14×14, **BA** is 21×21.

19. **AB** is not defined, **BA** is 4×2.

21. **AB** is not defined, **BA** is 7×6.

23. The adjacency matrix of G is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Compute

$$\mathbf{A}^3 = \begin{pmatrix} 2 & 7 & 7 & 4 & 4 \\ 7 & 8 & 9 & 9 & 9 \\ 7 & 9 & 8 & 9 & 9 \\ 4 & 9 & 9 & 6 & 7 \\ 4 & 9 & 9 & 7 & 6 \end{pmatrix} \text{ and } \mathbf{A}^4 = \begin{pmatrix} 14 & 17 & 17 & 18 & 18 \\ 17 & 34 & 33 & 26 & 26 \\ 17 & 33 & 34 & 26 & 26 \\ 18 & 26 & 26 & 25 & 24 \\ 18 & 26 & 26 & 24 & 25 \end{pmatrix}.$$

The number of $v_1 - -v_4$ walks of length 3 is $(\mathbf{A})^3_{14} = 4$ and the number of $v_1 - -v_4$ walks of length 4 is $(\mathbf{A}^4)_{14} = 18$. The number of $v_2 - -v_3$ walks of length 3 is 9 and the number of $v_2 - -v_4$ walks of length 4 is 26.

25. The adjacency matrix of K is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 11\ 01\ 1 & & & & \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Then

$$\mathbf{A}^2 = \begin{pmatrix} 4 & 2 & 3 & 3 & 2 \\ 2 & 3 & 2 & 2 & 3 \\ 3 & 2 & 4 & 3 & 2 \\ 3 & 2 & 3 & 4 & 2 \\ 2 & 3 & 2 & 2 & 3 \end{pmatrix}, \mathbf{A}^3 = \begin{pmatrix} 10 & 10 & 11 & 11 & 10 \\ 10 & 6 & 10 & 10 & 6 \\ 11 & 10 & 10 & 11 & 10 \\ 11 & 10 & 11 & 10 & 10 \\ 10 & 6 & 10 & 10 & 6 \end{pmatrix}$$

and

$$\mathbf{A}^4 = \begin{pmatrix} 42 & 32 & 41 & 41 & 32 \\ 32 & 30 & 32 & 32 & 30 \\ 41 & 32 & 42 & 41 & 32 \\ 41 & 32 & 31 & 42 & 32 \\ 32 & 30 & 32 & 32 & 30 \end{pmatrix}.$$

The number of $v_4 - -v_5$ walks of length 2 is 2 and the number of $v_2 - -v_3$ walks of length 3 is 10. The number of $v_1 - -v_2$ walks of length 4 is 32 and the number of $v_4 - -v_5$ walks of length 4 is 32.

11.2 Row Operations and Reduced Matrices

1. $\mathbf{A}$ is 3×4, so multiply $\mathbf{A}$ on the left by the matrix $\mathbf{\Omega}$ formed by performing this elementary row operation on $\mathbf{I}_3$. Thus form

$$\mathbf{\Omega} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This choice of $\mathbf{\Omega}$ can be checked by verifying that $\mathbf{\Omega A} = \mathbf{B}$.

3. In this case more than one elementary operation is performed, so Ω is a product of elementary operations, the right-most factor performing the first operation, then the next to right-most, the second operation, and so on, with the left factor performing the last operation:

$$\mathbf{\Omega} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \sqrt{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If these products are carried out, we get

$$\begin{pmatrix} 0 & 5 & 0 \\ 1 & 0 & \sqrt{13} \\ 0 & 0 & 1 \end{pmatrix}$$

5.

$$\mathbf{\Omega} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 15 \end{pmatrix} \begin{pmatrix} 1 & \sqrt{3} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 15 \\ 1 & \sqrt{3} \end{pmatrix}$$

7.

$$\mathbf{\Omega} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 14 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \\ 14 & 1 & 0 \end{pmatrix}$$

In these and later problems, the delta notation is sometimes useful:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

9. Let $\mathbf{A} = [a_{ij}]$ be an $n \times m$ matrix. Now, $\mathbf{B} = [b_{ij}]$ and $\mathbf{E} = [e_{ij}]$ are obtained, respectively, by interchanging rows s and t of $\mathbf{A}$ and $\mathbf{I}_n$. Then, for $i \neq s$ and $i \neq t$,
$$b_{ij} = a_{ij} \text{ and } e_{ij} = \delta_{ij}.$$
If $i = s$, $b_{sj} = a_{tj}$ and $e_{sj} = \delta_{tj}$. And if $i = t$, $b_{ij} = a_{sj}$ and $e_{ij} = \delta_{sj}$.

Now consider the $i, j --$ element of $\mathbf{EA}$. For $i \neq s$ and $i \neq t$,
$$(\mathbf{EA})_{sj} = \sum_{k=1}^{n} e_{ik}a_{kj} = a_{ij} = b_{ij}.$$
For $i = s$,
$$(\mathbf{EA})_{sj} = \sum_{k=1}^{n} e_{sk}a_{kj} = \sum_{k=1}^{n} \delta_{tk}a_{kj} = a_{tj} = b_{sj}.$$
And for $i = t$,
$$(\mathbf{EA})_{tj} = \sum_{k=1}^{n} e_{ik}a_{kj} = \sum_{k=1}^{n} \delta_{sk}a_{kj} = a_{sj} = b_{tj}$$
for $j = 1, \cdots, m$. Therefore $\mathbf{EA} = \mathbf{B}$.

11. Let $\mathbf{A}$ be $n \times m$. Now $\mathbf{B}$ and $\mathbf{E}$ are formed, respectively, from $\mathbf{A}$ and $\mathbf{I}_n$ by adding α times row s to row t. For $i \neq t$, $b_{ij} = a_{ij}$ and $e_{ij} = \delta_{ij}$, while for $i = t$, $b_{tj} = a_{tj} + \alpha a_{sj}$ and $e_{tj} = \delta_{tj} + \alpha\delta_{sj}$.

Now consider the $i, j --$ element of of $\mathbf{EA}$. For $i \neq t$,
$$(\mathbf{EA})_{ij} = \sum_{k=1}^{n} e_{ik}a_{kj} = \sum_{k=1}^{n} \delta_{ik}a_{kj} = a_{ij}.$$
For $i = t$,
$$\begin{aligned}(\mathbf{EA})_{tj} &= \sum_{k=1}^{n} e_{tk}a_{kj} = \sum_{k=1}^{n}(\delta_{tk} + \alpha a_{kj}) \\ &= a_{tj} + \alpha a_{sj} = b_{sj}.\end{aligned}$$
This shows that $\mathbf{EA} = \mathbf{B}$.

In Problems 12–23, keep in mind that a given matrix can be reduced by different sequences of row operations, but the end result must be the same - a matrix has only one reduced form. There may therefore be different matrices $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$ such that
$$\mathbf{\Omega}_1\mathbf{A} = \mathbf{A}_R = \mathbf{\Omega}_2\mathbf{A}.$$
We give one such matrix for each problem.

13. We can reduce $\mathbf{A}$ by first subtracting row two from row one of $\mathbf{I}_2$, then multiplying row one of the resulting matrix by 1/3:

$$\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1/3 & -1/3 \\ 0 & 1 \end{pmatrix} = \mathbf{\Omega}.$$

Then

$$\mathbf{\Omega A} = \mathbf{A}_R = \begin{pmatrix} 1 & 0 & 1/3 & 4/3 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

15. $\mathbf{A}$ is in reduced form, so $\mathbf{A} = \mathbf{A}_R$.

For Problems 17–23, just $\mathbf{\Omega}$ and $\mathbf{A}_R$ are given.

17.

$$\mathbf{\Omega} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}, \mathbf{A}_R = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

19.

$$\mathbf{\Omega} = \begin{pmatrix} -1/3 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{A}_R = \begin{pmatrix} 1 & -4/3 & -4/3 \\ 0 & 0 & 0 \end{pmatrix}$$

21.

$$\mathbf{\Omega} = \frac{1}{4}\begin{pmatrix} 0 & 0 & 1 \\ 4 & -4 & -8 \\ -4 & 8 & 8 \end{pmatrix}, \mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & -3/4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

23.

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 0 & -6 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \mathbf{A}_R = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

11.3 Solution of Homogeneous Linear Systems

In Problems 1–12, we use the facts that (1) the system $\mathbf{AX} = \mathbf{O}$ has the same solutions as the reduced system $\mathbf{A}_R\mathbf{X} = \mathbf{O}$, and (2) the solution of the reduced system can be read by inspection from the reduced matrix $\mathbf{A}_R$.

1. The coefficient matrix and its reduced form are

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix}, \mathbf{A}_R = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix}.$$

Because $\mathbf{A}_R$ has two nonzero rows and $m = 4$, the solution space has dimension $m - 2 = 2$, which means that the general solution is in terms of two of the unknowns, which can be given any values. Specifically, from the reduced matrix,

$$x_1 = -x_3 + x_4$$
$$x_2 = x_3 - x_4.$$

All solutions are given by

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_3 + x_4 \\ x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

It looks a little neater to write the general solution

$$\mathbf{X} = \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

in which α and β are arbitrary real numbers. The solution space of this system is the subspace of R^4 having basis vectors

$$< -1, 1, 1, 0 >, < 1, -1, 0, 1 > .$$

3. The coefficient matrix and its reduced form are

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}_3.$$

Here $\mathbf{A}$ has rank 3, the number of nonzero rows of $\mathbf{A}$, and $m-$ rank $(A) = 3 - 3 = 0$, so the solution space has dimension zero, consisting just of the trivial solution

$$\mathbf{X} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is consistent with the reduced system being the system

$$x_1 = 0, x_2 = 0, x_3 = 0.$$

5. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 3 & -1 & 4 \\ 2 & -2 & 1 & 1 & 0 \\ 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}$$

and

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & 0 & 9/4 \\ 0 & 1 & 0 & 0 & 7/4 \\ 0 & 0 & 1 & 0 & 5/8 \\ 0 & 0 & 0 & 1 & -13/8 \end{pmatrix}.$$

The solution space has dimension $5 - 4 = 1$ and the general solution is

$$\mathbf{X} = \alpha \begin{pmatrix} -9/4 \\ -7/4 \\ -5/8 \\ 13/8 \\ 1 \end{pmatrix}.$$

The single vector $< -9/4, -7/4, -5/8, 13/8, 1 >$ is a basis for the solution space.

7. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} -10 & -1 & 4 & 1 & 1 & -1 \\ 0 & 1 & -1 & 3 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

and

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & 05/6 & 5/9 & \\ 0 & 1 & 0 & 0 & 2/3 & 10/9 \\ 0 & 0 & 1 & 0 & 8/3 & 13/9 \\ 0 & 0 & 0 & 1 & 2/3 & 1/9 \end{pmatrix}.$$

From the reduced system read the general solution

$$\mathbf{X} = \alpha \begin{pmatrix} -5/6 \\ -2/3 \\ -8/3 \\ -2/3 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -5/9 \\ -10/9 \\ -13/9 \\ -1/9 \\ 0 \\ 1 \end{pmatrix}.$$

The solution space is a subspace of R^6 having dimension 2, with basis vectors

$$< -5/6, -2/3, -8/3, -2/3, 1, 0 >, < -5/9, -10/9, -13/9, -1/9, 0, 1 > .$$

9. There is no x_3 in the equations, so we actually have a system of three equations in the four unknowns x_1, x_2, x_4 and x_5. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & -3 & 1 \\ 2 & -1 & 1 & 0 \\ 2 & -3 & 0 & 4 \end{pmatrix}$$

and

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & -5/14\ 0 & 1 & 0 & -11/17 \\ 0 & 0 & 1 - 6/7 & & & & & \end{pmatrix}.$$

Now x_1, x_2 and x_4 depend on x_5 and

$$\mathbf{X} = \alpha \begin{pmatrix} 5/14 \\ 11/7 \\ 6/7 \\ 1 \end{pmatrix}.$$

The solution space is the subspace of R^4 of dimension 1, having basis vector $< 5/14, 11/7, 6/7, 1 >$.

11. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -2 & 3 \\ 1 & 0 & 0 & 0 & -1 & 2 & 0 \\ 2 & 0 & 0 & -3 & 1 & 0 & 0 \end{pmatrix}$$

and

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 & -1 & 3/2 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & -2/3 & 3 \\ 0 & 0 & 0 & 1 & -1 & 4/3 & 0 \end{pmatrix}.$$

The general solution is

$$\mathbf{X} = \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ -3/2 \\ 2/3 \\ -4/3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 1/2 \\ -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The solution space is the subspace of R^7 of dimension 3, with basis vectors

$< 1, 1, 0, 1, 1, 0, 0 >, < -2, -3/2, 2/3, -4/3, 0, 1, 0 >, < 0, 1/2, -3, 0, 0, 0, 1 >$.

13. The answer is yes. All that is required is that $m-$ rank$(\mathbf{A}) > 0$, so that the solution space has positive dimension, hence non-zero vectors, which are solutions of the system.

As a specific example, consider the system $\mathbf{AX} = \mathbf{O}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 3 & 0 & 9 \end{pmatrix}.$$

This is a homogeneous system with three equations in three unknowns. We find that

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\mathbf{A}_R$ has two nonzero rows, so the solution space of $\mathbf{AX} = \mathbf{O}$ has dimension $3 - 2 = 1 > 0$. In fact, the general solution is

$$\mathbf{X} = \alpha \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

15. Let the rows of $\mathbf{A}$ be $\mathbf{R}_1, \cdots, \mathbf{R}_n$. These are vectors in R^m. Let R be the row space of $\mathbf{A}$, which is the subspace of R^m spanned by the row vectors.

    Now, $\mathbf{X}$ is in the solution space $S(\mathbf{A})$ of the homogeneous system with coefficient matrix $\mathbf{A}$ exactly when $\mathbf{AX} = \mathbf{O}$. This is true exactly when the dot product $\mathbf{R}_j \cdot \mathbf{X} = \mathbf{O}$ for $j = 1, \cdots, n$, which is true exactly when each row is orthogonal to $\mathbf{X}$. But this is equivalent to $\mathbf{X}$ being orthogonal to every linear combination of these rows, hence to every vector in the row space R of $\mathbf{A}$. This makes the solution space of the system the orthogonal complement of the row space:

$$R^{\perp} = S(\mathbf{A}).$$

    Because the columns of $\mathbf{A}^t$ are the rows of $\mathbf{A}$, similar reasoning shows that the solution space $S(\mathbf{A}^t)$ of the system $\mathbf{A}^t\mathbf{X} = \mathbf{O}$ has the column space C of A as its orthogonal complement:

$$C^{\perp} = S(\mathbf{A}^t).$$

11.4 Nonhomogeneous Systems

1. The augmented matrix is

$$[\mathbf{A} \vdots \mathbf{B}] = \begin{pmatrix} 3 & -2 & 1 & \vdots & 6 \\ 1 & 10 & -1 & \vdots & 2 \\ -3 & -2 & 1 & \vdots & 0 \end{pmatrix}$$

    with reduced form

$$[\mathbf{A} \vdots \mathbf{B}]_R = \begin{pmatrix} 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & 1/2 \\ 0 & 0 & 1 & \vdots & 4 \end{pmatrix}.$$

    The reduced forms of the matrix of coefficients of the system and the augmented matrix have the same number of nonzero rows, so both $\mathbf{A}$ and $[\mathbf{A} \vdots \mathbf{B}]$ have the same rank (in this case 3). Therefore this system is consistent. We can read the solution from the reduced form of the augmented matrix:

$$\mathbf{X} = \begin{pmatrix} 1 \\ 1/2 \\ 4 \end{pmatrix}.$$

In this case the solution is unique because m minus the rank of $\mathbf{A}$ is $3 - 3 = 0$, so the associated homogeneous system has only the trivial solution.

3. We have

$$[\mathbf{A} \vdots \mathbf{B}] = \begin{pmatrix} 2 & -3 & 0 & 1 & 0 & -1 & \vdots & 0 \\ 3 & 0 & -2 & 0 & 1 & 0 & \vdots & 1 \\ 0 & 1 & 0 & -1 & 0 & 6 & \vdots & 3 \end{pmatrix}$$

and

$$[\mathbf{A} \vdots \mathbf{B}]_R = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 17/2 & \vdots & 9/2 \\ 0 & 1 & 0 & -1 & 0 & 6 & \vdots & 3 \\ 0 & 0 & 1 & -3/2 & -1/2 & 51/4 & \vdots & 25/4 \end{pmatrix}.$$

$\mathbf{A}$ and $[\mathbf{A} \vdots \mathbf{B}]$ have the same rank 3, so the system has solutions. Read from the reduced augmented matrix that

$$\mathbf{X} = \begin{pmatrix} 9/2 \\ 3 \\ 25/4 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 1 \\ 3/2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -17/2 \\ -6 \\ -51/4 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

5. The augmented matrix is

$$[\mathbf{A} \vdots \mathbf{B}] = \begin{pmatrix} 0 & 3 & 0 & -4 & 0 & 0 & \vdots & 10 \\ 1 & -3 & 0 & 0 & 4 & -1 & \vdots & 8 \\ 0 & 1 & 1 & -6 & 0 & 1 & \vdots & -9 \\ 1 & -1 & 0 & 0 & 0 & 1 & \vdots & 0 \end{pmatrix}$$

and its reduced form is

$$[\mathbf{A} \vdots \mathbf{B}]_R = \begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 2 & \vdots & -4 \\ 0 & 1 & 0 & 0 & -2 & 1 & \vdots & -4 \\ 0 & 0 & 1 & 0 & -7 & 9/2 & \vdots & -38 \\ 0 & 0 & 0 & 1 & -3/2 & 3/4 & \vdots & -11/2 \end{pmatrix}.$$

Both $\mathbf{A}$ and $[\mathbf{A} \vdots \mathbf{B}]$ have the same rank 4 (number of nonzero rows in their reduced forms), so this system has a solution. From the reduced augmented matrix we read the general solution

$$\mathbf{X} = \begin{pmatrix} -4 \\ -4 \\ -38 \\ -11/2 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 2 \\ 7 \\ 3/2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ -1 \\ -9/2 \\ -3/4 \\ 0 \\ 1 \end{pmatrix}.$$

7. The augmented matrix is

$$[\mathbf{A} \vdots \mathbf{B}] = \begin{pmatrix} 8 & -4 & 0 & 1/2 & 10 & \vdots & 1 \\ 0 & 1 & 0 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 1 & -3 & 2 & \vdots & 0 \end{pmatrix}$$

with reduced form

$$[\mathbf{A}_R \vdots \mathbf{C}] = \begin{pmatrix} 1 & 0 & 0 & 1/2 & 3/4 & \vdots & 9/8 \\ 0 & 1 & 0 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 1 & -3 & 2 & \vdots & 0 \end{pmatrix}.$$

The system is consistent (same number of zero rows in $\mathbf{A}_R$ and the reduced augmented matrix), and

$$\mathbf{X} = \begin{pmatrix} 9/8 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1/2 \\ -1 \\ 3 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -3/4 \\ 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}.$$

9. The augmented matrix is

$$\begin{pmatrix} 0 & 0 & 14 & 0 & -3 & 0 & 1 & \vdots & 2 \\ 1 & 1 & 1 & -1 & 0 & 1 & 0 & \vdots & -4 \end{pmatrix}$$

with reduced form

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 3/14 & 1 & -1/14 & \vdots & -29/7 \\ 0 & 0 & 1 & 0 & -3/14 & 0 & 1/14 & \vdots & 1/7 \end{pmatrix}.$$

The matrix of coefficients and its augmented matrix have rank 2, so the system is consistent. The general solution is

$$\mathbf{X} = \begin{pmatrix} -29/7 \\ 0 \\ 1/7 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -3/14 \\ 0 \\ 3/14 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} 1/14\ 0 \\ -1/14 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

11. The augmented matrix is

$$\begin{pmatrix} 7 & -3 & 4 & 0 & \vdots & -7 \\ 2 & 1 & -1 & 4 & \vdots & 6 \\ 0 & 1 & 0 & -3 & \vdots & -5 \end{pmatrix}$$

and the reduced augmented matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 19/15 & \vdots & 22/15 \\ 0 & 1 & 0 & -3 & \vdots & -5 \\ 0 & 0 & 1 & -67/13 & \vdots & -121/15 \end{pmatrix}.$$

The rank of $\mathbf{A}$ and of the augmented matrix is 3, so the system is consistent. The general solution is

$$\mathbf{X} = \begin{pmatrix} 22/15 \\ -5 \\ -121/15 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -19/15 \\ 3 \\ 67/15 \\ 1 \end{pmatrix}.$$

13. The augmented matrix is

$$\begin{pmatrix} 4 & -1 & 4 & \vdots & 1 \\ 1 & 1 & -5 & \vdots & 0 \\ -2 & 1 & 7 & \vdots & 4 \end{pmatrix}$$

and its reduced form is

$$\begin{pmatrix} 1 & 0 & 0 & \vdots & 16/57 \\ 0 & 1 & 0 & \vdots & 99/57 \\ 0 & 0 & 1 & \vdots & 23/57 \end{pmatrix}.$$

A and the augmented matrix both have rank 3, which is also the number of unknowns, so the system has a unique solution

$$\mathbf{X} = \begin{pmatrix} 16/57 \\ 99/57 \\ 23/57 \end{pmatrix}.$$

15. Write

$$\mathbf{X} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}$$

Let the columns of **A** be $\mathbf{C}_1, \cdots, \mathbf{C}_m$. Then $\mathbf{AX} = \mathbf{B}$ if and only if

$$\alpha_1 \mathbf{C}_1 + \cdots + \alpha_m \mathbf{C}_m = \mathbf{B}.$$

This means that **X** can be a solution if and only if **X** is a linear combination of the columns of **A**, hence is in the column space of **A**.

11.5 Matrix Inverses

Problem 1 shows all of the row operations used to reduce the matrix and find the inverse, or show that the matrix is singular. For Problems 2–10, just the inverse is given if the matrix is nonsingular.

1. Reduce

$$\begin{pmatrix} -1 & 2 & \vdots & 1 & 0 \\ 2 & 1 & \vdots & 0 & 1 \end{pmatrix} \rightarrow \text{ add 2 times row one to row two } \begin{pmatrix} -1 & 2 & \vdots & 1 & 0 \\ 0 & 5 & \vdots & 2 & 1 \end{pmatrix}$$

$$\rightarrow \text{ multiply row one by } -1 \begin{pmatrix} 1 & -2 & \vdots & -1 & 0 \\ 0 & 5 & \vdots & 2 & 1 \end{pmatrix}$$

$$\rightarrow \text{ multiply row two by } 1/5 \begin{pmatrix} 1 & -2 & \vdots & -1 & 0 \\ 0 & 1 & \vdots & 2/5 & 1/5 \end{pmatrix}$$

$$\rightarrow \text{ add 2 times row two to row one } \begin{pmatrix} 1 & 0 & \vdots & -1/5 & 2/5 \\ 0 & 1 & \vdots & 2/5 & 1/5 \end{pmatrix}.$$

Because $\mathbf{I}_2$ has appeared on the left, the given matrix is nonsingular and the right two columns of this augmented matrix form the inverse matrix:

$$\mathbf{A}^{-1} = \frac{1}{5}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}.$$

3.
$$\mathbf{A}^{-1} = \frac{1}{12}\begin{pmatrix} -2 & 2 \\ 1 & 5 \end{pmatrix}$$

5.
$$\mathbf{A}^{-1} = \frac{1}{12}\begin{pmatrix} 3 & -2 \\ -3 & 6 \end{pmatrix}$$

7.
$$\mathbf{A}^{-1} = \frac{1}{31}\begin{pmatrix} -6 & 11 & 2 \\ 3 & 10 & -1 \\ 1 & -7 & 10 \end{pmatrix}$$

9.
$$\mathbf{A}^{-1} = -\frac{1}{12}\begin{pmatrix} 6 & -6 & 0 \\ -3 & -9 & 2 \\ 3 & -3 & -2 \end{pmatrix}$$

11.
$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \frac{1}{11}\begin{pmatrix} -1 & -1 & 8 & 4 \\ -9 & 2 & -5 & 14 \\ 2 & 2 & -5 & 3 \\ 3 & 3 & -2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \\ -5 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} -23 \\ -75 \\ -9 \\ 14 \end{pmatrix}$$

13.
$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$
$$= \frac{1}{28}\begin{pmatrix} 11 & 12 & 9 \\ 3 & 16 & 5 \\ 8 & 24 & 4 \end{pmatrix}\begin{pmatrix} -4 \\ 5 \\ 8 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 22 \\ 27 \\ 30 \end{pmatrix}$$

15.
$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = -\frac{1}{25}\begin{pmatrix} 5 & -15 & -15 \\ -10 & 15 & 10 \\ -5 & 10 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -7 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -21 \\ 14 \\ 0 \end{pmatrix}$$

11.6 Determinants

In Problems 1–6 we provide a sequence of row and/or column operations leading to a determinant that is easily evaluated. Other sequences of operations can also be used.

1. Add 2 times row two to row one and then -7 times row two to row three to obtain

$$\begin{vmatrix} -2 & 4 & 1 \\ 1 & 6 & 3 \\ 7 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 16 & 7 \\ 1 & 6 & 3 \\ 0 & -42 & -17 \end{vmatrix} = (-1)^{3+1}(1)\begin{vmatrix} 16 & 7 \\ -42 & -17 \end{vmatrix} = -22.$$

3. Add column two to column one, then 2 times column two to column three:

$$\begin{vmatrix} -4 & 5 & 6 \\ -2 & 3 & 5 \\ 2 & -2 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 21 \\ 1 & 3 & 14 \\ 0 & -2 & 0 \end{vmatrix} = (-1)^{3+2}(-2)\begin{vmatrix} 1 & 21 \\ 1 & 14 \end{vmatrix} = -14.$$

5. Add 2 times column three to column one and then add column three to column two to obtain

$$\begin{vmatrix} 17 & -2 & 5 \\ 1 & 12 & 0 \\ 14 & 7 & -7 \end{vmatrix} = \begin{vmatrix} 27 & 3 & 5 \\ 1 & 12 & 0 \\ 0 & 0 & -7 \end{vmatrix} = (-1)^{3+3}(-7)\begin{vmatrix} 27 & 3 \\ 1 & 12 \end{vmatrix} = -2,247.$$

For Problems 7–10, we just give the value of the determinant.

7. -122

9. 72

11. $-15,698$

13. $3,372$

15. Add columns two, three and four to column one, then factor $(a+b+c+d)$ out of column one to obtain

$$\begin{vmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{vmatrix} = (a+b+c+d)\begin{vmatrix} 1 & b & c & d \\ 1 & c & d & a \\ 1 & d & a & b \\ 1 & a & b & c \end{vmatrix}$$

Now add

$$(-1)\text{row two } + \text{ row three } - \text{ row four}$$

to row one and factor out $b-a+d-c$ from the new row one to obtain

$$(a+b+c+d)(b-a+d-c)\begin{vmatrix} 1 & b & c & d \\ 1 & c & d & a \\ 1 & d & a & b \\ 1 & a & b & c \end{vmatrix}.$$

17. Use induction to prove that, for $n = 2, 3, \cdots$, the determinant of an $n \times n$ upper triangular matrix is the product of the main diagonal elements.

For $n = 2$, this is obvious because

$$\begin{vmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{vmatrix} = a_{11}a_{22}.$$

Now suppose the statement is true for some $n \geq 2$. We want to prove that it is true for $n+1$. Let $\mathbf{A}$ be an $n+1 \times n+1$ upper triangular matrix.

Then $a_{i1} = 0$ for $i = 2, \cdots, n+1$, so by expanding by column one, we have

$$|\mathbf{A}| = a_{11}|\mathbf{B}|,$$

where $\mathbf{B}$ is the $n \times n$ upper triangular matrix obtained by deleting row one and column one of $\mathbf{A}$. By the inductive hypothesis,

$$|\mathbf{B}| = a_{22}a_{33} \cdots a_{n+1,n+1}.$$

Then

$$|\mathbf{A}| = a_{11}a_{22} \cdots a_{n+1,n+1}.$$

This completes the proof by induction.

11.7 Cramer's Rule

1. $|\mathbf{A}| = 47 \neq 0$ so Cramer's rule applies:

$$x_1 = \frac{1}{47}\begin{vmatrix} 5 & -4 \\ -4 & 1 \end{vmatrix} = -\frac{11}{47}, x_2 = \frac{1}{47}\begin{vmatrix} 15 & 5 \\ 8 & -1 \end{vmatrix} = -\frac{100}{47}.$$

3. $|\mathbf{A}| = 132$ and the solution is

$$x_1 = \frac{1}{132}\begin{vmatrix} 0 & -4 & 3 \\ -5 & 5 & -1 \\ -4 & 6 & 1 \end{vmatrix} = -\frac{66}{132} = -\frac{1}{2},$$

$$x_2 = \frac{1}{132}\begin{vmatrix} 8 & 0 & 3 \\ 1 & -5 & -1 \\ -2 & -4 & 1 \end{vmatrix} = -\frac{114}{132} = -\frac{19}{22},$$

and

$$x_3 = \frac{1}{132}\begin{vmatrix} 8 & -4 & 0 \\ 1 & 5 & -5 \\ -2 & 6 & -4 \end{vmatrix} = \frac{24}{132} = \frac{2}{11}.$$

5. $|\mathbf{A}| = -6$ and

$$x_1 = \frac{5}{6}, x_2 = -\frac{10}{3}, x_3 = -\frac{5}{6}.$$

7. $|\mathbf{A}| = 4$ and the solution is

$$x_1 = -\frac{172}{2} = -86, x_2 = -\frac{109}{2}, x_3 = -\frac{43}{2}, x_4 = \frac{37}{2}.$$

9. $|\mathbf{A}| = 93$ and

$$x_1 = \frac{33}{93}, x_2 = -\frac{409}{93}, x_3 = -\frac{1}{93}, x_4 = \frac{116}{93}.$$

11.8 The Matrix Tree Theorem

1. The tree matrix for this graph is

$$\mathbf{T} = \begin{pmatrix} 2 & 0 & -1 & 0 & -1 \\ 0 & 2 & -1 & -1 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & -1 & 3 \end{pmatrix}.$$

Evaluate any 4×4 cofactor of $\mathbf{T}$ to obtain 21 as the number of spanning trees in the labeled graph.

3.

$$\mathbf{T} = \begin{pmatrix} 4 & -1 & 0 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & -1 & 0 \\ -1 & 0 & -1 & 4 & -1 & -1 \\ -1 & 0 & -1 & -1 & 3 & 0 \\ -1 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

and each cofactor gives 61 as the number of spanning trees

5.

$$\mathbf{T} = \begin{pmatrix} 3 & -1 & 0 & 0 & -1 & -1 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 4 & -1 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ -1 & -1 & -1 & -1 & 4 & 0 \\ -1 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

and the number of spanning trees is 61.

Chapter 12

Eigenvalues and Diagonalization

12.1 Eigenvalues and Eigenvectors

1.
$$p_{\mathbf{A}}(\lambda) = |\lambda \mathbf{I}_2 - \mathbf{A}| = \lambda^2 - 2\lambda - 5$$

so the eigenvalues of $\mathbf{A}$ are $1 + \sqrt{6}$ and $1 - \sqrt{6}$, with eigenvectors, respectively,

$$\mathbf{V}_1 = \begin{pmatrix} \sqrt{6} \\ 2 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} -\sqrt{6} \\ 1 \end{pmatrix}.$$

The Gerschgorin circles are of radius 3 about $(1, 0)$ and radius 2 about $(1, 0)$.

3. The characteristic equation is

$$\lambda^2 + 3\lambda - 10 = 0$$

and eigenvalues and eigenvectors are

$$\lambda_1 = -5, \mathbf{V}_1 = \begin{pmatrix} 7 \\ -1 \end{pmatrix}, \lambda_2 = 2, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

One Gerschgorin circle has radius 1 and center $(2, 0)$, and the other is the degenerate circle of radius 0 about $(-5, 0)$.

5. $p_{\mathbf{A}}(\lambda) = \lambda^2 - 3\lambda + 14$,

$$\lambda_1 = (3 + \sqrt{47}i)/2, \mathbf{V}_1 = \begin{pmatrix} -1 + \sqrt{47}i \\ 4 \end{pmatrix}$$

$$\lambda_2 = (3 - \sqrt{47}i)/2, \mathbf{V}_2 = \begin{pmatrix} -1 - \sqrt{47}i \\ 4 \end{pmatrix}$$

The Gerschgorin circles have radius 6, center $(1, 0)$, and radius 2, center $(2, 0)$.

7. $p_{\mathbf{A}}(\lambda) = \lambda^3 - 5\lambda^2 + 6\lambda$.

$$\lambda_1 = 0, \mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \lambda_2 = 2, \mathbf{V}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \lambda_3 = 3, \mathbf{V}_3 = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

The Gershgorin circle has radius 3, center $(0, 0)$.

9. $p_{\mathbf{A}}(\lambda) = \lambda^2(\lambda + 3)$

$$\lambda_1 = -3, \mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \lambda_2 = \lambda_3 = 0, \mathbf{V}_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

All eigenvectors associated with the double eigenvalue 0 are constant multiples of $\mathbf{V}_2$. The Gershgorin circle has radius 2, center $(-3, 0)$.

11. $p_{\mathbf{A}}(\lambda) = (\lambda + 14)(\lambda - 2)^2$,

$$\lambda_1 = -14, \mathbf{V}_1 = \begin{pmatrix} -16 \\ 0 \\ 1 \end{pmatrix} \lambda_2 = \lambda_3 = 2, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

All eigenvectors associated with λ_2 are constant multiples of $\mathbf{V}_2$. The Gershgorin circles have radius 1, center $(-14, 0)$ and radius 3, center $(2, 0)$.

13. $p_{\mathbf{A}}(\lambda) = \lambda(\lambda^2 - 8\lambda + 7)$,

$$\lambda_1 = 0, \mathbf{V}_1 = \begin{pmatrix} 14 \\ 7 \\ 10 \end{pmatrix}, \lambda_2 = 1, \mathbf{V}_2 = \begin{pmatrix} 6 \\ 0 \\ 5 \end{pmatrix}, \lambda_3 = 7, \mathbf{V}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The Gershgorin circles have radius 2, center $(1, 0)$, and radius 5, center $(7, 0)$.

15. $p_{\mathbf{A}}(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda^2 + \lambda - 13)$,

$$\lambda_1 = 1, \mathbf{V}_1 = \begin{pmatrix} -2 \\ -11 \\ 0 \\ 1 \end{pmatrix}, \lambda_2 = 2, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_3 = \frac{-1 + \sqrt{53}}{2}, \mathbf{V}_3 = \begin{pmatrix} \sqrt{53} - 7 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \lambda_4 = \frac{-1 - \sqrt{53}}{2}, \mathbf{V}_4 = \begin{pmatrix} -\sqrt{53} - 7 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

The Gershgorin circles have radius 2, center $(-4, 0)$ and radius 1, center $(3, 0)$.

17. We know that $\mathbf{AE} = \lambda\mathbf{E}$. Then

$$\begin{aligned}\mathbf{A}(\mathbf{AE}) &= \mathbf{A}^2\mathbf{E} = \mathbf{A}(\lambda\mathbf{E}) \\ &= \lambda\mathbf{AE} = \lambda(\lambda\mathbf{E}) = \lambda^2\mathbf{E}.\end{aligned}$$

This says that λ^2 is an eigenvalue of $\mathbf{A}^2$ with eigenvector $\mathbf{E}$. It is not a routine inductive argument to show that λ^n is an eigenvalue of $\mathbf{A}^n$ with eigenvector $\mathbf{E}$, for any positive integer n.

12.2 Diagonalization

1. The characteristic equation is $\lambda^2 - 3\lambda + 4 = 0$, so the eigenvalues are

$$\lambda_1 = \frac{3 + \sqrt{7}i}{2} \text{ and } \lambda_2 = \frac{3 - \sqrt{7}i}{2}.$$

Corresponding eigenvectors are

$$\mathbf{V}_1 = \begin{pmatrix} 2 \\ -3 + \sqrt{7}i \end{pmatrix} \text{ and } \mathbf{V}_2 = \begin{pmatrix} 2 \\ -3 - \sqrt{7}i \end{pmatrix}.$$

The matrix

$$\mathbf{P} = \begin{pmatrix} 2 & 2 \\ -3 + \sqrt{7}i & -3 - \sqrt{7}i \end{pmatrix}$$

diagonalizes $\mathbf{A}$, and

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} (3 + \sqrt{7}i)/2 & 0 \\ 0 & (3 - \sqrt{7}i)/2 \end{pmatrix}.$$

If we wrote the eigenvectors in different order in defining the columns of $\mathbf{P}$, then the columns of $\mathbf{P}^{-1}\mathbf{AP}$ would be reversed.

3. The characteristic equation is $\lambda^2 - 2\lambda + 1 = 0$, with repeated root 1. Every eigenvector is a scalar multiple of

$$\mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Because $\mathbf{A}$ does not have two independent eigenvectors, $\mathbf{A}$ is not diagonalizable.

5. The eigenvalues and corresponding eigenvectors are

$$\lambda_1 = 0, \mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \lambda_2 = 5, \mathbf{V}_2 = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}, \lambda_3 = -2, \mathbf{V}_3 = \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}.$$

The matrix

$$\begin{pmatrix} \mathbf{P} = 0 & 5 & 0 \\ 1 & 1 & -3 \\ 0 & 0 & 2 \end{pmatrix}$$

diagonalizes $\mathbf{A}$ and

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

7. Eigenvalues and corresponding eigenvectors are

$$\lambda_1 = 1, \mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \lambda_2 = \lambda_3 = -2, \mathbf{V}_2 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}.$$

All eigenvectors associated with the repeated eigenvalue -2 are scalar multiples of $\mathbf{V}_2$. Because $\mathbf{A}$ does not have three linearly independent eigenvectors, $\mathbf{A}$ is not diagonalizable.

9. The characteristic equation is

$$(\lambda - 1)(\lambda - 4)(\lambda^2 + 5\lambda + 5) = 0.$$

Eigenvalues are $\lambda_1 = 1, \lambda_2 = 4$, $\lambda_3 = (-5+\sqrt{5})/2$, and $\lambda_4 = (-5-\sqrt{5})/2)$. Corresponding eigenvectors are

$$\mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{V}_3 = \begin{pmatrix} 0 \\ (2 - 3\sqrt{5})/41 \\ (-1 + \sqrt{5})/2 \\ 1 \end{pmatrix}, \mathbf{V}_4 = \begin{pmatrix} 0 \\ (2 + 3\sqrt{5})/41 \\ (-1 - \sqrt{5})/2 \\ 1 \end{pmatrix}.$$

Let

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & (2 - 3\sqrt{5})/41 & (2 + 3\sqrt{5})/41 \\ 0 & 0 & (-1 + \sqrt{5})/2 & (-1 - \sqrt{5})/2 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then $\mathbf{P}$ diagonalizes $\mathbf{A}$:

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & & & \\ 0 & 1 & 0 & 00 & 0 & (-5 + \sqrt{5})/2 & 0 \\ 0 & 0 & 0 & (-5 - \sqrt{5})/2 & & & \end{pmatrix}.$$

11. Let

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & \\ 0 & \lambda_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Then

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P},$$

so

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}.$$

Then

$$\begin{aligned} \mathbf{A}^k &= (\mathbf{P}\mathbf{D}\mathbf{P}^{-1})^k \\ &= (\mathbf{P}\mathbf{D}\mathbf{P}^{-1})(\mathbf{P}\mathbf{D}\mathbf{P}^{-1} \cdots (\mathbf{P}\mathbf{D}\mathbf{P}^{-1}) \\ &= \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1}, \end{aligned}$$

with the interior pairings of $\mathbf{P}$ and $\mathbf{P}^{-1}$ canceling.

Problems 12–15 can be solved using the idea of Problem 11, coupled with the fact that the kth power of a diagonal matrix is the diagonal matrix formed by raising each diagonal element to the kth power.

13. Eigenvalues of $\mathbf{A}$ are $-1, -5$, and the matrix of respective eigenvectors,

$$\mathbf{P} = \begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix}$$

diagonalizes $\mathbf{A}$ to

$$\mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix}.$$

Now,

$$\mathbf{P}^{-1} = \begin{pmatrix} 1/4 & 0 \\ -1/4 & 1 \end{pmatrix}$$

so compute

$$\mathbf{A}^6 = \mathbf{P}\mathbf{D}^6\mathbf{P}^{-1} = \begin{pmatrix} 1 & 0 \\ -3906 & 15625 \end{pmatrix}.$$

15. Eigenvalues of $\mathbf{A}$ are $\sqrt{2}, -\sqrt{2}$. Form

$$\mathbf{P} = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{pmatrix}.$$

Then

$$\mathbf{P}^{-1} = \begin{pmatrix} \sqrt{2}/4 & 1/2 \\ -\sqrt{2}/4 & 1/2 \end{pmatrix}.$$

Let

$$\mathbf{B} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix}.$$

Then

$$\mathbf{A}^6 = \mathbf{P}\mathbf{D}^6\mathbf{P}^{-1} = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}.$$

12.3 Special Matrices and Their Eigenvalues and Eigenvectors

In Problems 1–12, find independent eigenvectors for the given matrix. Normalize these by dividing each eigenvector by its length. These normalized eigenvectors form columns of an orthogonal matrix that diagonalizes the given matrix.

It is routine to show that eigenvectors are orthogonal by taking their dot product. We will omit the arithmetic of this verification.

1. We find eigenvectors

$$\mathbf{V}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Divide each by its norm to get unit eigenvectors and form the orthogonal matrix

$$\mathbf{Q} = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}.$$

This is an orthogonal matrix that diagonalizes **A**.

3. Eigenvectors are

$$\mathbf{V}_1 = \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}.$$

Normalize these to form

$$\mathbf{Q} = \begin{pmatrix} \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} & \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{1}{\sqrt{4-2\sqrt{2}}} \end{pmatrix}.$$

5. Eigenvalues of **A** are 3, $\sqrt{2}-1$ and $-\sqrt{2}-1$, with corresponding eigenvectors

$$\mathbf{V}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 1 \\ \sqrt{2}-1 \\ 0 \end{pmatrix} \mathbf{V}_3 = \begin{pmatrix} 1 \\ -\sqrt{2}-1 \\ 0 \end{pmatrix}.$$

For an orthogonal matrix that diagonalizes $\mathbf{A}$, let

$$\mathbf{Q} = \begin{pmatrix} 0 & \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1}{\sqrt{4+2\sqrt{2}}} \\ 0 & \frac{\sqrt{2}-1}{\sqrt{4-2\sqrt{2}}} & \frac{-\sqrt{2}-1}{\sqrt{4+2\sqrt{2}}} \\ 1 & \frac{\sqrt{2}-1}{\sqrt{4-2\sqrt{2}}} & \frac{-\sqrt{2}-1}{\sqrt{4+2\sqrt{2}}} \end{pmatrix}.$$

7. Eigenvalues are
$$7, \frac{1}{2}(5+\sqrt{41}), \frac{1}{2}(5-\sqrt{41})$$
with corresponding eigenvectors
$$\mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 5+\sqrt{41} \\ 0 \\ 4 \end{pmatrix}, \mathbf{V}_3 = \begin{pmatrix} 5-\sqrt{41} \\ 0 \\ 4 \end{pmatrix}.$$
The following orthogonal matrix diagonalizes $\mathbf{A}$:
$$\mathbf{Q} = \begin{pmatrix} 1 & \frac{5+\sqrt{41}}{\sqrt{82+10\sqrt{41}}} & \frac{5-\sqrt{41}}{\sqrt{82-10\sqrt{41}}} \\ 1 & 0 & 0 \\ 0 & \frac{4}{\sqrt{82+10\sqrt{41}}} & \frac{4}{\sqrt{82-10\sqrt{41}}} \end{pmatrix}.$$

9. The matrix is not hermitian, skew-hermitian or unitary. Eigenvalues are $2, 2$.

11. The matrix is skew-hermitian because $\mathbf{S}^t = -\overline{\mathbf{S}}$. Eigenvalues are $0, \sqrt{3}i, -\sqrt{3}i$.

13. The matrix is not unitary, hermitian or skew-hermitian. Eigenvalues are $2, i, -i$.

15. Suppose $\mathbf{H}$ is hermitian. Then
$$\overline{\mathbf{H}} = \mathbf{H}^t.$$
Then
$$\overline{\mathbf{H}\mathbf{H}^t} = \overline{\mathbf{H}\mathbf{H}^t} = \overline{\mathbf{H}\overline{\mathbf{H}}} = \overline{\mathbf{H}}\mathbf{H}.$$

17. Suppose $\mathbf{S}$ is skew-hermitian. Then $\mathbf{S}^t = -\overline{\mathbf{S}}$, so
$$s_{jj} = -\overline{s_{jj}} \text{ for } j = 1, 2, \cdots, n.$$
Write $s_{jj} = a_{jj} + ib_{jj}$. Then
$$s_{jj} = a_{jj} + ib_{jj} = -\overline{a_{jj} + ib_{jj}} = -a_{jj} + ib_{jj}.$$
But then each $a_{jj} = -a_{jj}$, so $a_{jj} = 0$ for $j = 1, 2, \cdots, n$. This makes each diagonal element of $\mathbf{S}$ either pure imaginary (if $b_{jj} \neq 0$) or zero.

12.4 Quadratic Forms

1. The matrix of the quadratic form is

$$\mathbf{A} = \begin{pmatrix} -5 & 2 \\ 2 & 3 \end{pmatrix}.$$

This matrix has eigenvalues $-1 + 2\sqrt{5}, -1 - 2\sqrt{5}$ and the quadratic form has standard form

$$(-1 + 2\sqrt{5})y_1^2 + (-1 - 2\sqrt{5})y_2^2.$$

3. The matrix is

$$\begin{pmatrix} -3 & 2 \\ 2 & 7 \end{pmatrix}$$

with eigenvalues $2 \pm \sqrt{29}$. The standard form is

$$(2 + \sqrt{29})y_1^2 + (2 - \sqrt{29})y_2^2.$$

5. The matrix is

$$\begin{pmatrix} 0 & -3 \\ -3 & 4 \end{pmatrix}$$

with eigenvalues $2 \pm \sqrt{13}$. The standard form is

$$(2 + \sqrt{13})y_1^2 + (2 - \sqrt{13})y_2^2.$$

7. The matrix is

$$\begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$$

with eigenvalues $1 \pm \sqrt{2}$. the standard form is

$$(1 + \sqrt{2})y_1^2 + (1 - \sqrt{2})y_2^2.$$

Chapter 13

Systems of Linear Differential Equations

13.1 Linear Systems

1. The two given solutions are linearly independent because neither is a constant multiple of the other. Use them as columns of a fundamental matrix

$$\mathbf{\Omega}(t) = \begin{pmatrix} -e^{2t} & 3e^{6t} \\ e^{2t} & e^{6t} \end{pmatrix}.$$

Notice that $\mathbf{\Omega}(0)$ has a nonzero determinant, which is also an indicator that the columns are independent.

Now we have a general solution

$$\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C},$$

in which $\mathbf{C}$ is a 2×1 matrix of constants. To satisfy the initial condition, we need to choose $\mathbf{C}$ so that

$$\mathbf{\Omega}(0)\mathbf{C} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

This requires that

$$\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

Then $c_1 = 3, c_2 = 1$, so the solution of the initial value problem is

$$\mathbf{X}(t) = \begin{pmatrix} -e^{2t} & 3e^{6t} \\ e^{2t} & e^{6t} \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3e^{2t} + 3e^{6t} \\ 3e^{2t} + e^{6t} \end{pmatrix}.$$

3. Because $\mathbf{\Phi}(0)$ and $\mathbf{\Phi}(0)$ are independent in R^2, these solutions are independent. Form the fundamental matrix

$$\mathbf{\Omega}(t) = \begin{pmatrix} (2+2\sqrt{3})e^{(1+2\sqrt{3}t)} & (2-2\sqrt{3})e^{(1-2\sqrt{3})t} \\ e^{(1+2\sqrt{3})t} & e^{(1-2\sqrt{3})t} \end{pmatrix}.$$

Then $\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C}$ is a general solution. To solve the initial value problem, we need

$$\mathbf{\Omega}(0)\mathbf{C} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

and we find that we must choose

$$c_1 = 1 - \frac{1}{6}\sqrt{3}, c_2 = 1 + \frac{1}{6}\sqrt{3}.$$

5. Form the fundamental matrix

$$\mathbf{\Omega}(t) = \begin{pmatrix} e^t & -e^t & e^{-3t} \\ e^t & 0 & 3e^{-3t} \\ 0 & e^t & e^{-3t} \end{pmatrix}.$$

The general solution is $\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C}$. To solve the initial value problem, we need $\mathbf{C}$ such that

$$\mathbf{\Omega}(0)\mathbf{C} = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}.$$

Solve this equation to get

$$\mathbf{C} = \begin{pmatrix} 24 \\ 14 \\ -9 \end{pmatrix}.$$

13.2 Solution of $\mathbf{X}' = \mathbf{AX}$ When A Is Constant

In these problems, different fundamental matrices may be found, depending on the choice of eigenvectors used corresponding to eigenvalues of the coefficient matrix.

In each problem, the general solution has the form $\mathbf{X} = \mathbf{\Omega}(t)\mathbf{C}$, where $\mathbf{\Omega}(t)$ is a fundamental matrix. We will give one choice for $\mathbf{\Omega}(t)$.

1.

$$\mathbf{\Omega}(t) = \begin{pmatrix} 7e^{3t} & 0 \\ 5e^{3t} & e^{-4t} \end{pmatrix}$$

3.

$$\mathbf{\Omega}(t) = \begin{pmatrix} 1 & e^{2t} \\ -1 & e^{2t} \end{pmatrix}$$

5.

$$\mathbf{\Omega}(t) = \begin{pmatrix} 1 & 2e^{3t} & -e^{-4t} \\ 6 & 3e^{3t} & 2e^{-4t} \\ -13 & -2e^{3t} & e^{-4t} \end{pmatrix}$$

7. $\mathbf{A}$ has eigenvalues and eigenvectors

$$2+2i, \begin{pmatrix} 2i & 1 \end{pmatrix}, 2-2i, \begin{pmatrix} -2i \\ 1 \end{pmatrix}.$$

Write

$$\begin{pmatrix} 2i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

to form two independent solutions

$$\mathbf{\Phi}_1(t) = e^{2t} \begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix}$$

and

$$\mathbf{\Phi}_2(t) = e^{2t} \begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix}.$$

Use these as columns of a fundamental matrix

$$\mathbf{\Omega}(t) = \begin{pmatrix} -2\sin(2t) & 2\cos(2t) \\ \cos(2t) & \sin(2t) \end{pmatrix}.$$

9. $\mathbf{A}$ has eigenvalues $2, 5, 5$, with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -3 \\ 1 \end{pmatrix}.$$

All eigenvectors associated with 5 are scalar multiples of this eigenvector. Immediately we can write two independent solutions

$$\mathbf{\Phi}_1(t) = e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\mathbf{\Phi}_2(t) = e^{5t} \begin{pmatrix} -3 \\ -3 \\ 1 \end{pmatrix}.$$

For a third solution, denote the eigenvector associated with 5 as $\mathbf{E}$ and let

$$\mathbf{\Phi}_3(t) = \mathbf{E}te^{5t} + \mathbf{K}e^{5t}.$$

Substitute this into $\mathbf{X}' = \mathbf{AX}$ and use the fact that $\mathbf{AE} = \mathbf{E}$ to obtain

$$\mathbf{E} + 5\mathbf{K} = \mathbf{AK}.$$

If we let

$$\mathbf{K} = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

we obtain

$$\begin{aligned} -3a + 5b + 6c &= -3 \\ 3b + 9c &= -3 \\ -b - 3c &= 1. \end{aligned}$$

Then $a = -2/3, b = -1, c = 0$, so

$$\mathbf{K} = \begin{pmatrix} -2/3 \\ -1\ 0 \end{pmatrix}$$

and we obtain the third solution

$$\mathbf{\Phi}_3(t) = \begin{pmatrix} -3te^{5t} - (2/3)e^{5t} \\ -3te^{5t} - e^{5t} \\ te^{5t} \end{pmatrix}.$$

The three solutions obtained are linearly independent and can be used to form the columns of a fundamental matrix.

11. The coefficient matrix has eigenvalues $2+2i$ and $2-2i$, with corresponding eigenvectors

$$\begin{pmatrix} 2i \\ 1 \end{pmatrix}, \begin{pmatrix} -2i \\ 0 \end{pmatrix}.$$

From these form two independent solutions which for the columns of the fundamental matrix

$$\mathbf{\Omega}(t) = \begin{pmatrix} -2e^{2t}\sin(2t) & 2e^{2t}\cos(2t) \\ e^{2t}\cos(2t) & e^{2t}\sin(2t) \end{pmatrix}.$$

13. The coefficient matrix has eigenvalues $1 \pm i$. An eigenvector associated with $1+i$ is

$$\begin{pmatrix} 2+i \\ 1 \end{pmatrix}.$$

Form the fundamental matrix

$$\mathbf{\Omega}(t) = e^t \begin{pmatrix} 2\cos(t) - \sin(t) & \cos(t) + 2\sin(t) \\ \cos(t) & \sin(t) \end{pmatrix}.$$

15. The coefficient matrix has eigenvalues $-2, -1+2i, -1-2i$. From -2 we obtain the solution

$$\mathbf{\Phi}(t) = e^{-2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

An eigenvalue for $-1+2i$ is

$$\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + i \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}.$$

This gives us two more solutions:

$$\mathbf{\Phi}_2(t) = e^{-t} \begin{pmatrix} \cos(2t) \\ \cos(2t) - 2\sin(2t) \\ 3\cos(2t) \end{pmatrix}$$

and

$$\mathbf{\Phi}_3(t) = e^{-t} \begin{pmatrix} \sin(2t) \\ 2\cos(2t) + \sin(2t) \\ 3\sin(2t) \end{pmatrix}.$$

These solutions form the columns of a fundamental matrix.

17. The coefficient matrix has eigenvalues $3, 3$, and every eigenvector is a scalar multiple of

$$\mathbf{E} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

One solution is

$$\mathbf{\Phi}_1(t) = \begin{pmatrix} e^{3t} \\ 0 \end{pmatrix}.$$

Attempt a second solution

$$\mathbf{\Phi}_2(t) = \mathbf{E}te^{3t} + \mathbf{K}e^{3t}.$$

Solve for $\mathbf{K}$ to get

$$\mathbf{K} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}.$$

Then

$$\mathbf{\Phi}_2(t) = \begin{pmatrix} te^{3t} \\ e^{3t}/2 \end{pmatrix}.$$

These solutions form columns of a fundamental matrix.

19. The coefficient matrix has eigenvalues $4+\sqrt{29}i$ and $4-\sqrt{29}i$, with corresponding eigenvectors

$$\begin{pmatrix} -2 \\ 3 \end{pmatrix} + i \begin{pmatrix} -\sqrt{29} \\ 0 \end{pmatrix}.$$

This gives us two independent solutions:

$$\mathbf{\Phi}_1(t) = e^{4t} \begin{pmatrix} -2\cos(\sqrt{29}t) + \sqrt{29}\sin(\sqrt{29}t) \\ 3\cos(\sqrt{29}t) \end{pmatrix}$$

and

$$\mathbf{\Phi}_2(t) = e^{4t} \begin{pmatrix} -\sqrt{29}\cos(\sqrt{29}t) - 2\sin(\sqrt{29}t) \\ 3\sin(\sqrt{29}t) \end{pmatrix}.$$

21. The coefficient matrix has eigenvalues $1, 1, 3, 0$, with corresponding eigenvectors

$$\mathbf{V}_1 = \begin{pmatrix} 0 \\ -2 \\ -2 \\ 1 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{V}_3 = \begin{pmatrix} 3 \\ 2 \\ 2 \\ 0 \end{pmatrix}, \mathbf{V}_4 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

We can write a general solution

$$\mathbf{X}(t) = \mathbf{V}_1 e^t + \mathbf{V}_2 e^t + \mathbf{V}_3 e^{3t} + \mathbf{V}_4.$$

13.3 Exponential Matrix Solutions

In Problems 1–8, the exponential matrix can be obtained using the Putzer algorithm or a software program.

1.
$$e^{\mathbf{A}t} = \begin{pmatrix} \cos(2t) - \frac{1}{2}\sin(2t) & \frac{1}{2}\sin(2t) \\ -\frac{5}{2}\sin(2t) & \cos(2t) + \frac{1}{2}\sin(2t) \end{pmatrix}$$

3.
$$e^{\mathbf{A}t} = e^{3t} \begin{pmatrix} \cos(2t) + \sin(2t) & -\sin(2t) \\ 2\sin(2t) & \cos(2t) - \sin(2t) \end{pmatrix}$$

5.
$$e^{\mathbf{A}t} = e^{t} \begin{pmatrix} \cos(2t) & 2\sin(2t) \\ -\frac{1}{2}\sin(2t) & \cos(2t) \end{pmatrix}$$

7.
$$e^{\mathbf{A}t} = e^{-t/2} \begin{pmatrix} \cos(3\sqrt{3}t/2) + \frac{1}{\sqrt{3}}\sin(3\sqrt{3}t/2) & -\frac{2}{\sqrt{3}}\sin(3\sqrt{3}t/2) \\ \frac{2}{\sqrt{3}}\sin(3\sqrt{3}t/2) & \cos(3\sqrt{3}t/2) - \frac{1}{\sqrt{3}}\sin(3\sqrt{3}t/2) \end{pmatrix}$$

9. First, because $\mathbf{D}$ is a diagonal matrix, $\mathbf{D}^n$ is a diagonal matrix for any positive integer n, and the diagonal element of $\mathbf{D}^n$ is d_j^n. Then

$$e^{\mathbf{D}t} = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{D}^n t^n$$

and the diagonal element of $e^{\mathbf{D}t}$ is

$$\sum_{n=0}^{\infty} \frac{1}{n!}(d_j)^n t^n,$$

which is $e^{d_j t}$.

11. From the result of Problem 10,
$$e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1},$$
where $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, the diagonal matrix having the eigenvalues $\lambda_1, \cdots, \lambda_n$ of $\mathbf{A}$ down its diagonal. But from the result of Problem 9, $e^{\mathbf{D}t}$ is the diagonal matrix having diagonal elements $e^{\lambda_j t}$.

13.4 Solution of $\mathbf{X}' = \mathbf{AX} + \mathbf{G}$ for Constant $\mathbf{A}$

1. The coefficient matrix $\mathbf{A}$ has the repeated eigenvalue $3, 3$, and every eigenvector is a scalar multiple of
$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$
One solution of the homogeneous system $\mathbf{X}' = \mathbf{AX}$ is
$$e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$
Using methods from Section 13.2, find a second, independent solution of this homogeneous system to write the fundamental matrix
$$\mathbf{\Omega}(t) = e^{3t}\begin{pmatrix} 1+2t & 2t \\ -2t & 1-2t \end{pmatrix}.$$
For a particular solution of the nonhomogeneous system, first compute
$$\mathbf{\Omega}^{-1}(t) = e^{-3t}\begin{pmatrix} 1-2t & -2t \\ 2t & 1+2t \end{pmatrix}.$$
Now use variation of parameters to compute a solution of the nonhomogeneous system. For this method, we need
$$\begin{aligned} \mathbf{u}(t) &= \int \mathbf{\Omega}^{-1}(t)\mathbf{G}(t)\,dt \\ &= \int e^{3t}\begin{pmatrix} 1-2t & -2t \\ 2t & 1+2t \end{pmatrix}\begin{pmatrix} -3e^t \\ e^{3t} \end{pmatrix}\,dt \\ &= \int \begin{pmatrix} 6te^{-2t} - 3e^{-2t} - 2t \\ -6te^{-2t} + 1 + 2t \end{pmatrix}\,dt = \begin{pmatrix} -3te^{-2t} - t^2 \\ (3/2)(1+2t)e^{-2t} + t + t^2 \end{pmatrix}. \end{aligned}$$
The general solution is
$$\begin{aligned} \mathbf{X}(t) &= \mathbf{\Omega}(t)\mathbf{C} + \mathbf{\Omega}(t)\mathbf{u}(t) \\ &= e^{3t}\begin{pmatrix} 1+2t & 2t \\ -2t & 1-2t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &+ e^{3t}\begin{pmatrix} 1+2t & 2t \\ -2t & 1-2t \end{pmatrix}\begin{pmatrix} -3te^{-2t} - t^2 \\ (3/2)(1+2t)e^{-2t} + t + t^2 \end{pmatrix} \\ &= \begin{pmatrix} e^{3t}(c_1(1+2t) + 2c_2 t) + t^2 e^{3t} \\ e^{3t}(-2c_1 t + c_2(1-2t)) + (t-t^2)e^{3t} + 3e^t/2 \end{pmatrix}. \end{aligned}$$

3. **A** has eigenvalues $6, 6$ and eigenvectors are scalar multiples of

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We find the fundamental matrix

$$\mathbf{\Omega}(t) = \begin{pmatrix} 1 & 1+t \\ 1 & t \end{pmatrix}$$

for the associated homogeneous system. Use this and variation of parameters to find the general solution of the nonhomogeneous system:

$$\mathbf{X}(t) = e^{6t} \begin{pmatrix} c_1 + c_2(1+t) + 2t + t^2 - t^3 \\ c_1 + c_2 t + 4t^2 - t^3 \end{pmatrix}.$$

5. **A** has eigenvalues $1, 1, 3, 3$. The eigenvalue 3 has two independent eigenvectors

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -9 \\ 2 \\ 0 \end{pmatrix},$$

and 1 has the eigenvector

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

with all other eigenvectors associated with 1 scalar multiples of this one. A fundamental matrix for the homogeneous system $\mathbf{X}' = \mathbf{AX}$ is

$$\mathbf{\Omega}(t) = \begin{pmatrix} 0 & e^t & 0 & 0 \\ 0 & -2e^t e^t & -9e^{3t} & \\ 0 & 0 & 0 & 2e^{3t} \\ e^t & -5te^t & e^{3t} & 0 \end{pmatrix}.$$

The nonhomogeneous system has general solution

$$\mathbf{X}(t) = \begin{pmatrix} c_2 e^t \\ -2c_2 e^t + (c_3 - 9c_4)e^{3t} + e^t \\ 2c_4 e^{3t} \\ (c_1 - 5c_2 t)e^t + c_3 e^{3t} + (1+3t)e^t \end{pmatrix}.$$

For Problems 6–9, the idea is to find a general solution for the system, then solve for the constants to obtain the solution satisfying the initial condition. For these problems only the solution of the initial value problem is given.

7.

$$\mathbf{X}(t) = \begin{pmatrix} (-1 - 14t)e^t \\ (3 - 14t)e^t \end{pmatrix}$$

9.
$$\mathbf{X}(t) = \begin{pmatrix} (6 + 12t + (1/)t^2)e^{-2t} \\ (2 + 12t + (1/2)t^2)e^{-2t} \\ (3 + 38t + 66t^2 + (13/6)t^3)e^{-2t} \end{pmatrix}$$

13.5 Solution by Diagonalization

1. The coefficient matrix $\mathbf{A}$ is diagonalized by
$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$$
and we find that
$$\mathbf{P}^{-1} = \frac{1}{3}\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}.$$
The system for $\mathbf{Z}$ is
$$\mathbf{Z}' = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{Z} + \frac{1}{3}\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 10\cos(t) \end{pmatrix}.$$
This is the system
$$\mathbf{Z}'(t) = \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} -z_1 \\ 2z_2 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} -10\cos(t) \\ 10\cos(t) \end{pmatrix}.$$
Solve these two independent differential equations to get
$$\mathbf{Z}(t) = \begin{pmatrix} c_1e^{-t} - (5/3)\cos(t) - (5/3)\sin(t) \\ c_2e^{2t} - (4/3)\cos(t) + (2/3)\sin(t) \end{pmatrix}.$$
Then
$$\mathbf{X}(t) = \mathbf{PZ}(t) = \begin{pmatrix} c_1e^{-t} + c_2e^{2t} - 3\cos(t) - \sin(t) \\ c_1e^{-t} + 4c_2e^{2t} - 7\cos(t) + \sin(t) \end{pmatrix}.$$

3. The coefficient matrix has eigenvalues $0, 2$ and is diagonalized by
$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$
which has inverse
$$\mathbf{P}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$
The system for $\mathbf{Z}(t)$ has the solution
$$\mathbf{Z}(t) = \begin{pmatrix} c_1 - 2t + e^{3t} \\ c_2e^{2t} - 1 + 3e^{3t} \end{pmatrix}.$$
Then
$$\mathbf{X}(t) = \begin{pmatrix} c_1 + 2c_2e^{2t} - 1 - 2t + 4e^{3t} \\ -c_1 + c_2e^{2t} - 1 + 2t + 2e^{3t} \end{pmatrix}.$$

5. $\mathbf{A}$ has eigenvalues $3i, -3i$ and is diagonalized by
$$\mathbf{P} = \begin{pmatrix} 1+i & 1-i \\ 3 & 3 \end{pmatrix}.$$
We find that
$$\mathbf{P}^{-1} = \frac{1}{6}\begin{pmatrix} -3i & 1+i \\ 3i & 1-i \end{pmatrix}.$$
The transformed problem for $\mathbf{Z}$ has the solution
$$\mathbf{Z}(t) = \begin{pmatrix} d_1 e^{3it} + ((2-i)/6)e^{2t} \\ d_2 e^{-3it} + ((2+i)/6)e^{2t} \end{pmatrix}.$$
If Euler's formula is used on the complex exponential terms, we obtain the real solution
$$\mathbf{X}(t) = \begin{pmatrix} c_1(\cos(3t) - \sin(3t)) - c_2(\cos(3t) + \sin(3t)) + e^{2t} \\ 3c_1 \cos(3t) 0 3c_2 \sin(3t) + 2e^{2t} \end{pmatrix}.$$

7. The coefficient matrix has eigenvalues $0, 3$ and is diagonalized by
$$\mathbf{P} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}.$$
Compute
$$\mathbf{P}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}.$$
The uncoupled system is
$$\mathbf{Z}' = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}\mathbf{Z} + \begin{pmatrix} 1/2 & 1/3 \\ -1/3 & 2/3 \end{pmatrix}\begin{pmatrix} 2t \\ 5 \end{pmatrix}.$$
The initial condition is
$$\mathbf{Z}(0) = \mathbf{P}^{-1}\mathbf{X}(0) = \begin{pmatrix} 25/3 \\ 11/3 \end{pmatrix}.$$
Then
$$\mathbf{Z}(t) = \begin{pmatrix} (1/3)t^2 + (5/3)t + 25/3 \\ (127/27)e^{3t} + (2/9)t - 28/27 \end{pmatrix}.$$
The solution of the initial value problem for $\mathbf{X}$ is
$$\mathbf{X}(t) = \mathbf{PZ}(t) = \begin{pmatrix} -(127/27)e^{3t} + (2/3)t^2 + (28/9)t + 478/27 \\ (127/27)e^{3t} + (1/3)t^2 + (17/9)t + 197/27 \end{pmatrix}.$$

9. The coefficient matrix has eigenvalues $1, 1, -3$, but there are two independent eigenvectors associated with the repeated eigenvalue 1 and $\mathbf{A}$ is diagonalized by
$$\mathbf{P} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix}.$$

We find that

$$\mathbf{P}^{-1} = \begin{pmatrix} 3 & -2 & 3 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{pmatrix}.$$

With $\mathbf{X} = \mathbf{PZ}$ we obtain the uncoupled system

$$\mathbf{Z}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix} \mathbf{Z} + \begin{pmatrix} 3 & =2 & 3 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -3e^{-3t} \\ t \\ 0 \end{pmatrix}.$$

The initial conditions are

$$\mathbf{Z}(0) = \mathbf{P}^{-1}\mathbf{X}(0) = \begin{pmatrix} 11 \\ 6 \\ -4 \end{pmatrix}.$$

Solve this uncoupled system to obtain

$$\mathbf{X}(t) = \mathbf{PZ}(t) = \begin{pmatrix} (5/2)e^{t} - (8/3)e^{-3t} + 3te^{-3t} + (8/9) + (4/3)t \\ (27/4)e^{t} - (113/12)e^{-3t} + 9te^{-3t} + (5/3) + 3t \\ (17/4)e^{t} - (113/36)e^{-3t} + 3te^{-3t} + (8/9) + (4/3)t \end{pmatrix}.$$

Chapter 14

Nonlinear Systems and Qualitative Analysis

14.1 Nonlinear Systems and Phase Portraits

1. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 3 & -5 \\ 5 & -7 \end{pmatrix},$$

with eigenvalues $-2, -2$. Every eigenvector is a scalar multiple of

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The origin is an improper nodal sink.

For Problems $3 - -16$, only the eigenvalues of the coefficient matrix, and the classification of the origin, are given. As typical cases, phase portraits are drawn for the systems of Problems $3, 5, 6, 7$ and 11.

Phase portraits are included for the systems of Problems $3, 4, 5$ and 11.

3. eigenvalues $2i, -2i$; center

5. $4 + 5i, 4 - 5i$, spiral point

7. $3, 3$, and the coefficient matrix does not have two independent eigenvectors; improper node

9. eigenvalues $-2 + \sqrt{3}i, -2 - \sqrt{3}i$, spiral sink

11. $\sqrt{5}, -\sqrt{5}$, saddle point

13. $-3 + \sqrt{7}, -3 - \sqrt{7}$, both eigenvalues negative, nodal sink

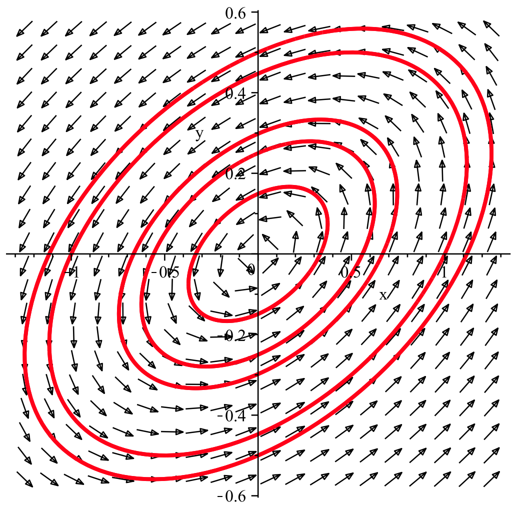

Figure 14.1: Center of Problem 3.

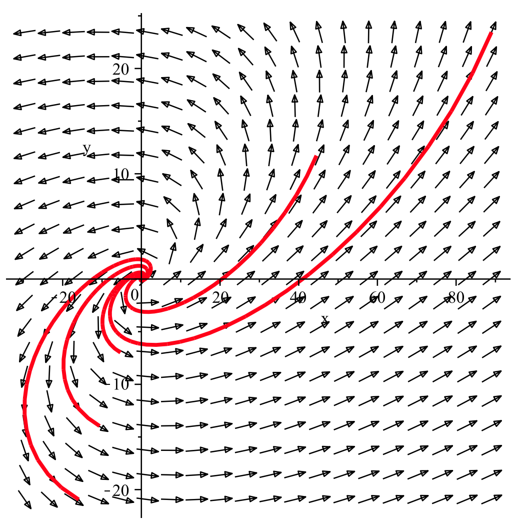

Figure 14.2: Spiral source of Problem 5.

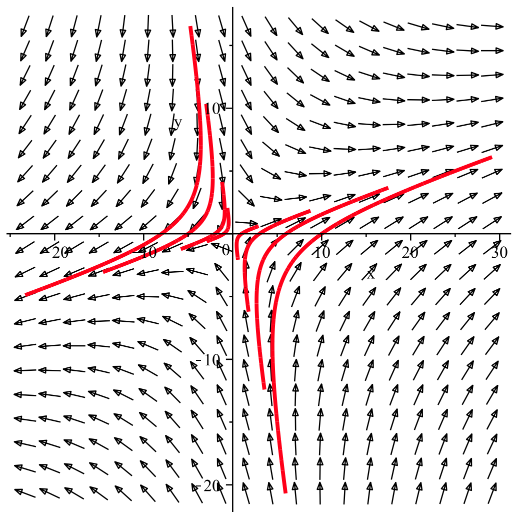

Figure 14.3: Saddle point of Problem 11.

15. $2+\sqrt{3}, 2-\sqrt{3}$, nodal source

17. (a) First, write

$$\frac{dx}{dt} = \frac{1}{t}x$$

as

$$\frac{1}{x}\,dx = \frac{1}{t}\,dt.$$

Integrate to get

$$\ln(x) = \ln(t) + c$$

for $x > 0, t > 0$. Then $x = ct$ with c constant. Put this into the second equation to get

$$y' = ct - \frac{1}{t}y.$$

Write this as

$$y' + \frac{1}{t}y = ct,$$

or

$$ty' + y = ct^2.$$

Then

$$(ty)' = ct^2.$$

Integrate to get

$$ty = \frac{c}{3}t^3 + d.$$

Then

$$y = \frac{c}{3}t^2 + \frac{d}{t}.$$

(b) Suppose $x(t_0) = 1$ and $y(t_0) = 0$. Then it is routine to solve for c and d from part (a) to obtain

$$x(t) = \frac{1}{t_0}t, y(t) = \frac{1}{3t_0}t^2 - \frac{1}{3t}t_0^2.$$

(c) In part (b), we have trajectories through $(1, 0)$ at any time $t_0 \neq 0$. However, these translations are not trajectories of each other.

14.2 Critical Points and Stability

In Problems 1–16, the stability type of the origin is given, based on information in Problems 1–16 of Section 14.1.

1. The origin is an improper node that is both stable and asymptotically stable

3. stable but not asymptotically stable center

5. unstable spiral source

7. unstable improper node

9. stable and asymptotically stable spiral sink

11. unstable saddle point

13. stable and asymptotically stable nodal sink

15. unstable nodal source

17. If $\epsilon = 0$, the eigenvalues are $\sqrt{5}i, -\sqrt{5}i$ and the origin is a center, which is stable but not asymptotically stable.

    If $\epsilon > 0$, the eigenvalues are

    $$\frac{1}{2}\epsilon + \frac{1}{2}\sqrt{(\epsilon-2)^2 - 24}, \frac{1}{2}\epsilon\sqrt{(\epsilon-2)^2 - 24}.$$

    These have positive real part. If $0 < \epsilon < 2(1+\sqrt{6})$, then the origin is an unstable spiral point. If $\epsilon > 2(1+\sqrt{6})$, the origin is an unstable saddle point. If $\epsilon = 2(1+\sqrt{6})$, the origin is an unstable improper node.

14.3 Almost Linear Systems

In Problem 1, the details of showing that the system is almost linear are included. Problems 2–10 omit this demonstration and concentrate on analyzing the critical point $(0,0)$.

1. To show that the system is almost linear, consider

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} &= \lim_{r\to 0} \frac{r^2\cos^2(\theta)}{r} \\ &= \lim_{r\to 0} r\cos^2(\theta) = 0.\end{aligned}$$

   The origin is a critical point and the matrix of coefficients of the linear part is

$$\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix},$$

   which has eigenvalues

$$\frac{1}{2}(3+\sqrt{3}i) \text{ and } \frac{1}{2}(3-\sqrt{3})i.$$

   The origin is an unstable spiral point of the linear part, hence also of the given system.

3. The linear part has matrix

$$\mathbf{A}_{(0,0)} = \begin{pmatrix} -2 & 2 \\ 1 & 4 \end{pmatrix}$$

   with eigenvalues $1+\sqrt{11}, 1-\sqrt{11}$. These are of opposite sign, so the origin is an unstable saddle point.

5. The linear part has matrix

$$\begin{pmatrix} 3 & 12 \\ -1 & -3 \end{pmatrix},$$

   with eigenvalues $\sqrt{3}i, -\sqrt{3}i$. The origin is a center for the linear part of the system, so the nonlinear system could have a center or a spiral point there.

7. The linear part has matrix

$$\begin{pmatrix} -3 & -4 \\ 1 & 1 \end{pmatrix}.$$

   This matrix has eigenvalues $-1, -1$, and all eigenvectors are scalar multiples of

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

   The origin is a stable improper nodal sink.

9. The linear part has matrix

$$\begin{pmatrix} -2 & -1 \\ -4 & 1 \end{pmatrix}$$

with eigenvalues $2, -3$, so the origin is an unstable saddle point.

11. Refer to these as systems I and II, in the order given.

(a) For each system the linear part has coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with eigenvalues $i, -i$. Therefore the origin is a center for each system.

(b) For The first system, use polar coordinates, with

$$x = r\cos(\theta), y = r\sin(\theta)$$

and

$$\sqrt{x^2 + y^2} = r.$$

Then

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{-x\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} &= \lim_{r\to 0} \frac{-r^2\cos(\theta)}{r} \\ &= \lim_{r\to 0} -r\cos(\theta) = 0, \end{aligned}$$

independent of θ. And, similarly

$$\lim_{(x,y)\to(0,0)} \frac{-y\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \lim_{r\to 0} -r\sin(\theta) = 0.$$

Therefore the first system is almost linear. The argument is essentially the same for the second system.

(c) With $r^2 = x^2 + y^2$, we have

$$rr' = xx' + yy',$$

where primes denote differentiation with respect to t. Now insert the expressions for x' and y' from the differential equations of system I to obtain

$$\begin{aligned} rr' &= x(y - x\sqrt{x^2+y^2}) + y(-x - y\sqrt{x^2+y^2}) \\ &= -(x^2+y^2)\sqrt{x^2+y^2} \\ &= -r^3. \end{aligned}$$

Then

$$r' = \frac{dr}{dt} = -r^2 \text{ for system I.}$$

Similarly, if we insert the expressions for x' and y' from system II, we obtain

$$\begin{aligned} rr' &= x(y + x\sqrt{x^2+y^2}) + y(-x + y\sqrt{x^2+y^2}) \\ &= (x^2+y^2)\sqrt{x^2+y^2} \\ &= r^3. \end{aligned}$$

Then

$$\frac{dr}{dt} = r^2 \text{ for system II.}$$

(d) For system I,

$$\frac{dr}{dt} = -r^2.$$

This is the separable equation

$$-\frac{1}{r^2}\,dr = dt.$$

This shows that, for system I, $r'(t) < 0$, so the distance between the point and the origin is decreasing with time.

Now integrate the differential equation for $r(t)$ to get

$$\frac{1}{r} = t + c.$$

To satisfy the initial condition $r(t_0) = r_0$, we need

$$\frac{1}{r_0} = t_0 + c,$$

so

$$c = \frac{1}{r_0} - t_0.$$

Then

$$r(t) = \frac{1}{t - t_0 + 1/r_0}$$

for system I. Then

$$r(t) \to 0 \text{ as } t \to \infty.$$

The first system is asymptotically stable at the origin.

(e) By an entirely analogous derivation, for system II,

$$r' = r^2 > 0$$

so $r(t)$ is increasing with time for system II. Solve this separable differential equation subject to $r(t_0) = r_0$ to get

$$r(t) = \frac{1}{1/r_0 + t_0 - t}.$$

Then for system II, $r(t) \to \infty$ as $t \to t_0 + 1/r_0$ from the left. We conclude from this that the second system is unstable at the origin.

Parts (d) and (e) show that, in the case of a center at the origin, behavior of the the linear part of an almost linear system at the origin does not provide definitive information about the stability of the origin for the nonlinear system.

13. Using results from Problem 11, we have

$$\begin{aligned} rr' &= xx' + yy' \\ &= x(y + \epsilon x(x^2 + y^2)) + y(-x + \epsilon y(x^2 + y^2)) \\ &= \epsilon(x^2 + y^2)(x^2 + y^2) \\ &= \epsilon r^4. \end{aligned}$$

Then

$$\frac{dr}{dt} = \epsilon r^3.$$

This is separable

$$\frac{1}{r^3}\, dr = \epsilon\, dt.$$

Integrate to get

$$-\frac{1}{2} r^{-2} = \epsilon t + c.$$

Then

$$r(t) = \frac{1}{\sqrt{k - 2\epsilon t}},$$

where $k = 2c$ is an arbitrary constant which is determined by specifying a point that the trajectory is to pass through at some positive time.

If $\epsilon < 0$, then

$$r(t) = \frac{1}{\sqrt{k + 2|\epsilon| t}} \to 0$$

as $t \to \infty$. In this case trajectories approach the origin as $t \to \infty$ and the nonlinear system is asymptotically stable.

However, something different happens if $\epsilon > 0$. Suppose $r(0) = \rho$, so a trajectory starts at a point ρ units from the origin at time zero. Then $k = 1/\rho^2$ and

$$r(t) = \frac{1}{\sqrt{(1/\rho)^2 - 2\epsilon t}}.$$

Now, as t starts at zero and increases toward $1/(2\epsilon\rho^2)$, $r(t) \to \infty$, so there is a time close to which the point is arbitrarily far from the origin. This makes the origin unstable.

14.4 Linearization

1. For critical points other than the origin, solve

$$x - y + x^2 = 0, x + 2y = 0$$

to get $(-3/2, 3/4)$. We find that

$$\mathbf{A}_{(-3/2,3/4)} = \begin{pmatrix} -2 & -1 \\ 1 & 2 \end{pmatrix},$$

which has eigenvalues $\sqrt{3}$ and $-\sqrt{3}$. This critical point is an unstable saddle point of the linear part, and therefore of the given system.

3. The critical point other than the origin is $(-5, -5)$. We find that

$$\mathbf{A}_{(-5,-5)} = \begin{pmatrix} -2 & 2 \\ 1 & -6 \end{pmatrix}.$$

This has eigenvalues $-4 + \sqrt{6}, -4 - \sqrt{6}$, which are unequal and both negative. This critical point is a stable and asymptotically stable nodal sink of the nonlinear system.

5. The system has one critical point other than the origin, $(-1/2, 1/8)$. Calculate

$$\mathbf{A}_{(-1/2,1/8)} = \begin{pmatrix} 3 & 12 \\ -1/4 & -3 \end{pmatrix},$$

with eigenvalues are $\sqrt{6}, -\sqrt{6}$, so this critical point is an unstable saddle point.

7. Aside from the origin, the system has critical points $(1/2, -1/2)$. We have

$$\mathbf{A}_{(1/2,-1/2)} = \begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix}$$

with eigenvalues $(-1 \pm \sqrt{3}i)/2$, so $(1/2, -1/2)$ is a spiral point, stable and asymptotically stable because the real part of the eigenvalues is negative.

9. The critical point other than the origin is $(-3/8, -3/2)$. We find that

$$\mathbf{A}_{(-3/8,-3/2)} = \begin{pmatrix} -2 & -2 \\ -4 & 1 \end{pmatrix}$$

with eigenvalues $(-1 \pm \sqrt{23}i)/2$, so $(-3/8, -3/2)$ is a stable and asymptotically stable spiral point.

Chapter 15

Vector Differential Calculus

15.1 Vector Functions of One Variable

In Problems 1 and 2 the details of the differentiation are carried out both ways. For Problems 3–8 just the derivative is given.

1. First use the product rule:

$$\begin{aligned}(f(t)\mathbf{F}(t))' &= f'(t)\mathbf{F}(t) + f(t)\mathbf{F}'(t)\\ &= (-12\sin(3t))\mathbf{F}(t) + 4\cos(3t)(6t\mathbf{j} + 2\mathbf{k})\\ &= -12\sin(3t)\mathbf{i} + (24t\cos(3t) - 36t^2\sin(3t))\mathbf{j}\\ &+ (8\cos(3t) - 24t\sin(3t))\mathbf{k}.\end{aligned}$$

   If we first carry out the product, we have

$$f(t)\mathbf{F}(t) = 4\cos(3t)\mathbf{i} + 12t^2\cos(3t)\mathbf{j} + 8t\cos(3t)\mathbf{k},$$

   so

$$\begin{aligned}&(f(t)\mathbf{F}(t))' = -12\sin(3t)\mathbf{i}\\ &+ (24t\cos(3t) - 36t^2\sin(3t))\mathbf{j} + (8\cos(3t) - 24t\sin(3t))\mathbf{k}.\end{aligned}$$

3. Apply the product rule for cross products:

$$\begin{aligned}(\mathbf{F}(t)\times\mathbf{G}(t))' &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & 0\\ 1 & -\cos(t) & t\end{vmatrix} + \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ t & 1 & 4\\ 0 & \sin(t) & 1\end{vmatrix}\\ &= -t\mathbf{j} - \cos(t)\mathbf{k} + (1 - 4\sin(t))\mathbf{i} - t\mathbf{j} + t\sin(t)\mathbf{k}\\ &= (1 - 4\sin(t))\mathbf{i} - 2t\mathbf{j} - (\cos(t) - t\sin(t))\mathbf{k}.\end{aligned}$$

5.

$$(f(t)\mathbf{F}(t))' = (1-8t^2)\mathbf{i} + (6t^2\cosh(t) - (1-2t^3)\sinh(t))\mathbf{j} + (e^t - 6t^2e^t - 2t^3e^t)\mathbf{k}$$

7.

$$(\mathbf{F}(t)\times\mathbf{G}(t))' = te^t(2+t)(\mathbf{j}-\mathbf{k})$$

9.

$$\mathbf{F}(t) = \sin(t)\mathbf{i} + \cos(t)\mathbf{j} + 45t\mathbf{k}$$

is a position vector for the curve, and

$$\mathbf{F}'(t) = \cos(t)\mathbf{i} - \sin(t)\mathbf{j} + 45\mathbf{k}$$

is a tangent vector. The distance function along C is

$$s(t) = \int_0^t \parallel \mathbf{F}'(\tau) \parallel \, d\tau = \int_0^t \sqrt{2026}\, d\tau = \sqrt{2026}t.$$

Then $t = s/\sqrt{2026}$ and we can write a position vector in terms of s:

$$\mathbf{G}(s) = \mathbf{F}(t(s)) = \sin\left(\frac{s}{\sqrt{2026}}\right)\mathbf{i} + \cos\left(\frac{s}{\sqrt{2026}}\right)\mathbf{j} + \frac{45s}{\sqrt{2026}}\mathbf{k}.$$

Now we can write a tangent vector in terms of s:

$$\mathbf{G}'(s) = \frac{1}{\sqrt{2026}}\left[\cos\left(\frac{s}{\sqrt{2026}}\right)\mathbf{i} - \sin\left(\frac{s}{\sqrt{2026}}\right)\mathbf{j} + 45\mathbf{k}\right].$$

11. $\mathbf{F} = t^2(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})$ is a position vector for the curve, and

$$\mathbf{F}'(t) = 2t(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})$$

is a tangent vector. The distance function along the curve is given by

$$s(t) = \int_1^t \parallel \mathbf{F}'(\xi) \parallel \, d\xi = 2\sqrt{29}\int_1^t \xi\, d\xi = \sqrt{29}(t^2-1).$$

Then

$$t = \sqrt{1 + s/\sqrt{29}}.$$

Let

$$\mathbf{G}(s) = \mathbf{F}(t(s)) = \left(\frac{s}{\sqrt{29}} + 1\right)(2\mathbf{i}+3\mathbf{j}+4\mathbf{k}).$$

Then

$$\mathbf{G}'(s) = \frac{1}{\sqrt{29}}(2\mathbf{i}+3\mathbf{j}+\mathbf{k})$$

is a unit tangent vector.

15.2 Velocity, Acceleration and Curvature

In Problems 1–10, we can compute

$$\mathbf{v} = \mathbf{F}'(t), \mathbf{a}(t) = \mathbf{F}''(t), v(t) = \parallel \mathbf{v}(t) \parallel$$

by straightforward calculations. Next,

$$\mathbf{T}(t) = \frac{1}{v(t)}\mathbf{v}(t) = \frac{1}{\parallel \mathbf{F}'(t) \parallel}\mathbf{F}'(t).$$

Tangential and normal components of the acceleration can be obtained as

$$a_T = \frac{dv}{dt} \text{ and } a_N = \sqrt{\parallel \mathbf{a} \parallel^2 - a_T^2}.$$

The unit normal is

$$\mathbf{N}(t) = \frac{1}{a_N}(\mathbf{a}(t) - a_T\mathbf{T}(t)).$$

In this way it is not necessary to compute $s(t)$ and write vectors in terms of s, which is often awkward. We can also compute

$$\mathbf{N}(t) = \frac{1}{\parallel d\mathbf{T}/dt \parallel}\frac{d\mathbf{T}}{dt}.$$

Curvature is often easily computed as

$$\kappa = \frac{a_N}{v^2}.$$

We can also compute curvature as

$$\kappa = \frac{\parallel \mathbf{T}'(t) \parallel}{\parallel \mathbf{F}'(t) \parallel}.$$

κ can also be obtained from a formula requested in Problem 13:

$$\kappa = \frac{\parallel \mathbf{F}'(t) \times \mathbf{F}''(t) \parallel}{\parallel \mathbf{F}'(t) \parallel^3}.$$

1. The velocity is

   $$\mathbf{v}(t) = \mathbf{F}'(t) = 3\mathbf{i} + 2t\mathbf{k}$$

   and the speed is

   $$v(t) = \parallel \mathbf{v}(t) \parallel = \sqrt{9 + 4t^2}.$$

   The acceleration is

   $$\mathbf{a}(t) = \mathbf{F}''(t) = 2\mathbf{k}.$$

   A unit tangent is

   $$\mathbf{T}(t) = \frac{1}{\sqrt{9 + 4t^2}}(3\mathbf{i} + 2t\mathbf{k}).$$

The curvature is

$$\kappa = \frac{\| \mathbf{T}'(t) \|}{\| \mathbf{F}'(t) \|} = \frac{6}{(9+4t^2)^{3/2}}.$$

Finally,

$$a_T = \frac{dv}{dt} = \frac{4t}{\sqrt{9+4t^2}}$$

and

$$a_N = \sqrt{\| \mathbf{a} \|^2 - a_T^2} = \frac{6}{\sqrt{9+4t^2}}.$$

3.

$$\mathbf{v}(t) = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}, v = 3,$$

$$\mathbf{T} = \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k}),$$

$$a_T = a_N = \kappa = 0$$

5.

$$\mathbf{v}(t) = -3e^{-t}(\mathbf{i} + \mathbf{j} - 2\mathbf{k}), \mathbf{a}(t) = 3e^{-t}(\mathbf{i} + \mathbf{j} - 2\mathbf{k}),$$

$$v(t) = 3\sqrt{6}e^{-t}, \mathbf{T}(t) = \frac{1}{\sqrt{6}}(-\mathbf{i} - \mathbf{j} + 2\mathbf{k}),$$

$$a_T = -3\sqrt{6}e^{-t}, a_N = 0, \kappa = 0$$

7.

$$\mathbf{v}(t) = 2\cosh(t)\mathbf{j} - 2\sinh(t)\mathbf{k}, v(t) = 2\sqrt{\cosh(2t)},$$

$$\mathbf{a}(t) = 2\sinh(t)\mathbf{j} - 2\cosh(t)\mathbf{k},$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{\cosh(t)}}(\cosh(t)\mathbf{j} - \sinh(t)\mathbf{k}),$$

$$\mathbf{a}(t) = \frac{2\sinh(2t)}{\sqrt{\cosh(2t)}}, a_N = \frac{2}{\sqrt{\cosh(2t)}},$$

$$\kappa = \frac{1}{2(\cosh(2t))^{3/2}}$$

Here we have used the identity

$$\cosh(2t) = \cosh^2(t) + \sinh^2(t).$$

9.

$$\mathbf{v}(t) = 2t(\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}),$$

$$\mathbf{a}(t) = 2(\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k})$$

$$v(t) = 2|t|\sqrt{\alpha^2 + \beta^2 + \gamma^2},$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}}(\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k})$$

$$a_N = 0, \kappa = 0,$$

and

$$a_N = 2(\text{sgn}(t))\sqrt{\alpha^2 + \beta^2 + \gamma^2},$$

where

$$\text{sgn}(t) = \begin{cases} 1 & \text{if } t > 0, \\ -1 & \text{if } t < 0. \end{cases}$$

11. The position vector for a straight line has the form

$$\mathbf{F}(t) = (a + bt)\mathbf{i} + (c + dt)\mathbf{j} + (p + ht)\mathbf{k}.$$

The tangent vector is the constant vector

$$\mathbf{T}(t) = b\mathbf{i} + d\mathbf{j} + h\mathbf{k}.$$

Then $\mathbf{T}'(t) = \mathbf{O}$, so $\kappa = 0$.

Conversely, suppose C is a smooth curve having zero curvature. Then

$$\kappa = \| \mathbf{T}'(s) \| = \| \mathbf{F}''(s) \| = 0.$$

If we write

$$\mathbf{F}(s) = f(s)\mathbf{i} + g(s)\mathbf{j} + h(s)\mathbf{k},$$

this means that

$$f''(s) = g''(s) = h''(s) = 0.$$

But then $f(s) = a + bs, g(s) = c + ds, h(s) = p + hs$ for some constants a, b, c, d, p, h. This makes $\mathbf{F}(s)$ the position vector of a straight line.

13. First write

$$\mathbf{T}(t) = \frac{1}{\| \mathbf{F}'(t) \|}\mathbf{f}'(t) = \frac{1}{v(t)}\mathbf{F}'(t).$$

This enables us to write

$$\mathbf{F}' = v\mathbf{T}.$$

Now, $\mathbf{F}''(t)$ is the acceleration $\mathbf{a}(t)$, and $\mathbf{T} \times \mathbf{T} = \mathbf{O}$, so

$$\begin{aligned} v\mathbf{T} \times \mathbf{F}'' &= v\mathbf{T}(a_T\mathbf{T} + a_N\mathbf{N}) \\ &= va_T\mathbf{T} \times \mathbf{T} + va_N\mathbf{T} \times \mathbf{N} \\ &= va_N\mathbf{T} \times \mathbf{N} \\ &= v(v^2\kappa)\mathbf{T} \times \mathbf{N}. \end{aligned}$$

But $\mathbf{T}$ and $\mathbf{N}$ are orthogonal unit vectors, so

$$\| \mathbf{T} \times \mathbf{N} \| = 1.$$

Then

$$\| \mathbf{F}' \times \mathbf{F}'' \| = v^3\kappa.$$

Finally,

$$v = \| \mathbf{F}' \|$$

so

$$\kappa = \frac{\| \mathbf{F}'(t) \times \mathbf{F}''(t) \|}{\| \mathbf{F}'(t) \|}^3.$$

15.3 The Gradient Field

1.

$$\nabla\varphi(x,y,z) = \frac{\partial}{\partial x}(xyz)\mathbf{i} + \frac{\partial}{\partial y}(xyz)\mathbf{j} + \frac{\partial}{\partial z}(xyz)\mathbf{k} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k},$$

$$\nabla\varphi(1,1,1) = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

The maximum value of $D_{\mathbf{u}}(1,1,1)$ is

$$\| \nabla\varphi(1,1,1) \| = \sqrt{3}.$$

The minimum value is $-\sqrt{3}$.

3.

$$\nabla\varphi(x,y,z) = (2y + e^z)\mathbf{i} + 2x\mathbf{j} + xe^z\mathbf{k},$$

$$\nabla\varphi(-2,1,6) = (2 + e^6)\mathbf{i} - 4\mathbf{j} - 2e^6\mathbf{k}.$$

The maximum value of $D_{\mathbf{u}}(-2,1,6)$ is

$$\sqrt{20 + 4e^6 + 5e^{12}}$$

and the minimum value is the negative of this.

5.

$$\nabla\varphi(x,y,z) = 2y\sinh(2xy)\mathbf{i} + 2x\sinh(2xy)\mathbf{j} - \cosh(z)\mathbf{k},$$

$$\nabla\varphi(0,1,1) = -\cosh(1)\mathbf{k}.$$

The maximum value of $D_{\mathbf{u}}(0,1,1)$ is $\cosh(1)$. The minimum value is $-\cosh(1)$.

7.

$$\begin{aligned}
D_{\mathbf{u}}\varphi(x,y,z) &= \nabla\varphi(x,y,z) \cdot \mathbf{u} \\
&= ((8y^2 - z)\mathbf{i} + 16xy\mathbf{j} - x\mathbf{k}) \cdot \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k}) \\
&= \frac{1}{\sqrt{3}}(8y^2 - z + 16xy - x)
\end{aligned}$$

9.

$$
\begin{aligned}
&D_{\mathbf{u}}(x,y,z)\\
&\qquad = (2xyz^3\mathbf{i} + x^2z^3\mathbf{j} + 3x^2yz^2\mathbf{k}) \cdot \frac{1}{\sqrt{5}}(2\mathbf{j} + \mathbf{k})\\
&\qquad = \frac{1}{\sqrt{5}}(2x^2z^3 + 3x^2yz^2)
\end{aligned}
$$

11. Let $\varphi(x,y,z) = x^2 + y^2 + z^2$ so the level surface is given by $\varphi(x,y,z) = 4$. The gradient provides a normal vector

$$\mathbf{N}(x,y,z) = \nabla\varphi(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}.$$

Then

$$\mathbf{N}(1,1,\sqrt{2}) = 2\mathbf{i} + 2\mathbf{j} + 2\sqrt{2}\mathbf{k}$$

is normal to the surface at $(1,1,\sqrt{2})$. The tangent plane at this point has the equation

$$2(x-1) + 2(y-1) + 2\sqrt{2}(z-\sqrt{2}) = 0,$$

or

$$x + y + \sqrt{2}z = 4.$$

The normal line at $(1,1,\sqrt{2})$ has parametric equations

$$x = y = 1 + 2t, z = \sqrt{2}(1+2t)$$

for all real t.

13. Let $\varphi(x,y,z) = x^2 - y^2 - z^2$. The normal vector at $(1,1,0)$ is

$$\mathbf{N}(1,1,0) = \nabla\varphi(1,1,0) = 2\mathbf{i} - 2\mathbf{j}.$$

The tangent plane at $(1,1,0)$ has equation

$$2x - 2y = 0$$

or $x = y$. The normal line at $(1,1,0)$ has parametric equations

$$x = 1 + 2t, y = 1 - 2t, z = 0.$$

15. A normal vector is given by

$$\mathbf{N} = \nabla(2x - \cos(xyz))\Big|_{(1,\pi,1)} = 2\mathbf{i}.$$

The tangent plane has equation $x = 1$ and the normal line at the point has parametric equations

$$x = 1 + 2t, y = \pi, z = 1.$$

17. Because $\nabla\varphi(x,y,z) = \mathbf{i} + \mathbf{k}$, the normal to the surface $\varphi(x,y,z) = c$ is the constant vector

$$\mathbf{N}(x,y,z) = \mathbf{i} + \mathbf{k}.$$

The surface must therefore be the plane $x + z = c$.

15.4 Divergence and Curl

1.

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(2z) = 4,$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & 2z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{O},$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

3.

$$\nabla \cdot \mathbf{F} = 2y + xe^y + 2,$$

$$\nabla \times \mathbf{F} = (e^y - 2x)\mathbf{k},$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial}{\partial x}(e^y - 2x) = 0.$$

5.

$$\nabla \cdot \mathbf{F} = \cosh(x) + xz\sinh(xyz) - 1,$$

$$\nabla \times \mathbf{F} = (-1 - xy\sinh(xyz))\mathbf{i} - \mathbf{j} + yz\sinh(xyz)\mathbf{k},$$

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{F}) &= (-y + y)\sinh(xyz) \\ &\quad + ((-xy^2z + xy^2z)\cosh(xyz) = 0.\end{aligned}$$

7.

$$\nabla\varphi = \mathbf{i} - \mathbf{j} + 4z\mathbf{k},$$

$$\nabla \times (\nabla\varphi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & -1 & 4z \end{vmatrix} = 0.$$

9.

$$\nabla\varphi = -6x^2yz^2\mathbf{i} - 2x^3z^2\mathbf{j} - 4x^3yz\mathbf{k},$$

$$\begin{aligned}\nabla \times (\nabla\varphi) &= (-4x^3z + 4x^3z)\mathbf{i} \\ &\quad + (-12x^2yz\mathbf{i} + 12x^2yz)\mathbf{j} + (6x^2z^2 - 6x^2z^2)\mathbf{k} = \mathbf{O}.\end{aligned}$$

11.

$$\begin{aligned}\nabla\varphi &= (\cos(x+y+z) - x\sin(x+y+z))\mathbf{i} \\ &\quad - x\sin(x+y+z)\mathbf{j} - x\sin(x+y+z)\mathbf{k},\end{aligned}$$

$$\begin{aligned}\nabla \times (\nabla\varphi) &= (-x\cos(x+y+z) + x\cos(x+y+z))\mathbf{i} \\ &+ (-\sin(x+y+z) - x\cos(x+y+z) + \sin(x+y+z) + x\cos(x+y+z))\mathbf{j} \\ &+ (-\sin(x+y+z) - x\cos(x+y+z) + \sin(x+y+z) + x\cos(x+y+z))\mathbf{k} \\ &= \mathbf{O}.\end{aligned}$$

13. Let $\mathbf{F} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$. Then

$$\begin{aligned}\nabla \cdot (\varphi \mathbf{F}) &= \nabla \cdot (\varphi f\mathbf{i} + \varphi g\mathbf{j} + \varphi h\mathbf{k}) \\ &= \frac{\partial}{\partial x}(\varphi f) + \frac{\partial}{\partial y}(\varphi g) + \frac{\partial}{\partial z}(\varphi h) \\ &= \varphi_x f + \varphi_y g + \varphi_z h \\ &+ \varphi(f_x + g_y + h_z) \\ &= \nabla\varphi \cdot \mathbf{F} + \varphi(\nabla \cdot \mathbf{F}).\end{aligned}$$

Next,

$$\nabla \times (\varphi \mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \varphi f & \varphi g & \varphi h \end{vmatrix}$$
$$= \left[\frac{\partial}{\partial y}(\varphi h) - \frac{\partial}{\partial z}(\varphi g)\right]\mathbf{i}$$
$$+ \left[\frac{\partial}{\partial z}(\varphi f) - \frac{\partial}{\partial x}(\varphi h)\right]\mathbf{j}$$
$$+ \left[\frac{\partial}{\partial x}(\varphi g) - \frac{\partial}{\partial y}(\varphi f\right]\mathbf{k}$$
$$= \left[\frac{\partial\varphi}{\partial y}h - \frac{\partial\varphi}{\partial z}g\right]\mathbf{i} + \left[\frac{\partial\varphi}{\partial z}f - \frac{\partial\varphi}{\partial x}h\right]\mathbf{j} + \left[\frac{\partial\varphi}{\partial x}g - \frac{\partial\varphi}{\partial y}f\right]\mathbf{k}$$
$$+ \varphi\left[\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right]\mathbf{i} + \left[\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right]\mathbf{j} + \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right]\mathbf{k}$$
$$= \nabla\varphi \times \mathbf{F} + \varphi(\nabla \times \mathbf{F}).$$

15.5 Streamlines of a Vector Field

1. The streamlines satisfy
$$dx = -\frac{dy}{y^2} = \frac{dz}{z}.$$
Integrate $dx = -(1/y^2)\,dy$ to obtain
$$x = \frac{1}{y} + c_1.$$
Next integrate $dx = (1/z)\,dz$ to get
$$x = \ln|z| + c_2.$$
Using x as the parameter, we can write equations of the streamline for this vector field:
$$x = x, y = \frac{1}{x - c_1}, z = e^{x - c_2}.$$

For the streamline through $(2,1,1)$, let $x=2$. Then

$$1 = \frac{1}{2-c_1} \text{ and } 1 = e^{2-c_2}.$$

Then$c_1 = 1$ and $c_2 = 2$, so this streamline has parametric equations

$$x = x, y = \frac{1}{x-1}, z = e^{x-2}.$$

3. We have

$$x\,dx = \frac{dy}{e^x} = \frac{dz}{-1}.$$

Integrate $xe^x\,dx = dy$ to obtain

$$y = xe^x - e^x + c_1.$$

Integrate $x\,dx = -dz$ to get

$$x^2 = -2z + c_2.$$

Using x as parameter, streamlines are given by

$$y = xe^x - e^x + c_1, z = \frac{1}{2}(c_2 - x^2).$$

For the streamline through $(2,0,4)$, we need

$$e^2 + c_1 = 0 \text{ and } 4 = \frac{1}{2}(c_2 - 4).$$

Then $c_1 = -e^2$ and $c_2 = 12$, so this streamline has parametric equations

$$x = x, y = xe^x - e^x - e^2, z = \frac{1}{2}(12 - x^2).$$

5. Streamlines satisfy

$$\frac{dy}{2e^z} = -\frac{dz}{\cos(y)}.$$

This is the separable equation

$$\cos(y)\,dy = -2e^z\,dz.$$

Integrate this to get

$$\sin(y) = c_2 - 2e^z.$$

We also have $x = c_1$. For the streamline through $3, \pi/4, 0)$, we need $c_1 = 3$ and $c_2 = 2 + \sqrt{2}/2$. With y as parameter, this streamline is given by

$$x = 3, y = y, z = \ln\left(\frac{\sqrt{2}}{4} + 1 - \frac{1}{2}\sin(y)\right).$$

7. Circular streamlines about the origin in the $x, y--$plane can be written as $x^2 + y^2 = r^2$, so

$$x\,dx + y\,dy = 0.$$

Then

$$\frac{dx}{y} = -\frac{dy}{x}, dz = 0.$$

A vector field having these streamlines is

$$\mathbf{F}(x, y) = \frac{1}{x}\mathbf{i} - \frac{1}{y}\mathbf{j}.$$

Chapter 16

Vector Integral Calculus

16.1 Line Integrals

1. On C, $x = t, y = t$ and $z = t^3$, so

$$\begin{aligned}\int_C x\,dx - dy + z\,dz &= \int_0^1 (t(1) - (1) + t^3(3t^2))\,dt \\ &= \int_0^1 (t - 1 + 3t^5)\,dt = 0\end{aligned}$$

3.

$$\begin{aligned}\int_C (x+y)\,ds &= \int_0^2 (2t\sqrt{1+1+4t^2}\,dt \\ &= \int_0^2 2t\sqrt{2+4t^2}\,dt = \frac{1}{6}(2+4t^2)^{3/2}\Big|_0^2 = \frac{26\sqrt{2}}{3}\end{aligned}$$

5.

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{R} &= \int_0^3 (\cos(t)\mathbf{i} + t^2\mathbf{j} + t\mathbf{k})\cdot(\mathbf{i} - 2t\mathbf{j} + 0\mathbf{k})\,dt \\ &= \int_0^3 (\cos(t) - 2t^3)\,dt = \sin(3) - \frac{81}{2}\end{aligned}$$

7. Parametrize C as $x = 2\cos(t), y = 2\sin(t), z = 0$ for $0 \le t \le 2\pi$. Then

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{R} &= \int_0^{2\pi} (2\cos(t)\mathbf{i} + 2\sin(t)\mathbf{j})\cdot(-2\sin(t)\mathbf{i} + 2\cos(t)\mathbf{j})\,dt \\ &= \int_0^{2\pi} (-4\cos(t)\sin(t) + 4\cos(t)\sin(t))\,dt = 0.\end{aligned}$$

9.

$$\int_C -xyz\,dz = \int_4^9 -z\sqrt{z}\,dz$$
$$= -\frac{2}{5}z^{5/2}\Big|_4^9 = -\frac{422}{5}$$

11. Parametrize the line segment as

$$x = y = z = 1 + 3t \text{ for } 0 \le t \le 1.$$

The work done is

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_0^1 ((1+3t)^2 - 2(1+3t)^2 + 1 + 3t)(3)\,dt$$
$$= \left[\frac{(1+3t)^2}{2} - \frac{(1+3t)^3}{3}\right]_0^1 = -\frac{27}{2}.$$

13. Take $\mathbf{F}(x) = f(x)\mathbf{i}$ and $\mathbf{R}(t) = t\mathbf{j}$ for $a \le t \le b$. The graph of the curve is defined by this position vector is the interval $[a, b]$, and

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_a^b f(x)\,dx.$$

16.2 Green's Theorem

1. The work done by $\mathbf{F}$ is

$$\begin{aligned} \text{work} &= \oint_C xy\,dx + x\,dy = \iint_\Omega \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(xy)\right] dA \\ &= \int_0^1 \int_0^{6x} (1-x)\,dy\,dx + \int_1^4 \int_0^{8-2x} (1-x)\,dy\,dx \\ &= \int_0^1 6x(1-x)\,dx + \int_1^4 (8-2x)(1-x)\,dx = -8 \end{aligned}$$

3.

$$\begin{aligned} \text{work} &= \oint_C (-\cosh(4x^4) + xy)\,dx + (e^{-y} + x)\,dy \\ &= \iint_D \left[\frac{\partial}{\partial x}(e^{-y} + x) - \frac{\partial}{\partial y}(-\cosh(4x^4) + xy)\right] dA \\ &= \iint_D (1-x)\,dA = \int_1^3 \int_1^7 (1-x)\,dy\,dx \\ &= \int_1^3 6(1-x)\,dx = -12 \end{aligned}$$

5.

$$\begin{aligned}
&\oint_C \mathbf{F} d\dot{\mathbf{R}} \\
&= \iint_D \left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2)\right] dA \\
&= \iint_D (-2y)\, dA = \int_1^6 \int_{(y+4)/5}^{(22-2y)/5} -2y\, dx\, dy \\
&= \int_1^6 (3y-18)\frac{2y}{5}\, dy = -40
\end{aligned}$$

7.

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_D \frac{\partial}{\partial x}(8xy^2)\, dA = \iint_D 8y^2\, dA.$$

To evaluate this integral, change to polar coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$, with $0 \le \theta \le 2\pi$ and $0 \le r \le 4$. We get

$$\begin{aligned}
\iint_D 8y^2\, dA &= \int_0^{2\pi}\int_0^4 8r^2\sin^2(\theta) r\, dr\, d\theta \\
&= \int_0^{2\pi} \sin^2(\theta)\, d\theta \int_0^4 8r^3\, dr = 512\pi.
\end{aligned}$$

9.

$$\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{R} &= \iint_D \left[\frac{\partial}{\partial x}(-e^x\sin(y)) - \frac{\partial}{\partial y}(e^x\cos(y))\right] dA \\
&= \iint_D (-e^x\sin(y) + e^x\sin(y))\, dA = 0
\end{aligned}$$

11.

$$\begin{aligned}
\oint_C \mathbf{F} d\dot{\mathbf{R}} &= \iint_D \left[\frac{\partial}{\partial x}(xy^2 - e^{cos(y)}) - \frac{\partial}{\partial y}(xy)\right] dA \\
&= \iint_D (y^2 - x)\, dA = \int_0^3 \int_0^{5-5x/3} (y^2 - x)\, dy\, dx \\
&= \int_0^3 \frac{1}{3}\left(5 - \frac{5x}{3}\right)\, dx - \int_0^3 x\left(5 - \frac{5x}{3}\right)], dx \\
&= \frac{95}{4}
\end{aligned}$$

13. By Green's theorem,

$$\begin{aligned}
\oint_C -\frac{\partial}{\partial y}\, dx + \frac{\partial}{\partial x}\, dy &= \iint_D \left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)\right] dA \\
&= \iint_D \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] dA.
\end{aligned}$$

15. If C does not enclose the origin, then Green's theorem applies and

$$\begin{aligned}
&\oint_C \mathbf{F} \cdot d\mathbf{R} \\
&= \iint_D \left[\frac{\partial}{\partial x} \left(\frac{x}{x^2+y^2} - 2y \right) - \frac{\partial}{\partial y} \left(\frac{-y}{x^2+y^2} + x^2 \right) \right] dA \\
&= \iint_D 0 \, dA = 0.
\end{aligned}$$

If C does enclose the origin, let K be a circle about the origin of sufficiently small radius r that K is in the region enclosed by C. Then, using the extended Green's theorem and polar coordinates, we have

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{R} &= \oint_K \mathbf{F} \cdot d\mathbf{R} \\
&= \int_0^{2\pi} \left[\left(\frac{-r\sin(\theta)}{r^2} + r^2\cos^2(\theta) \right) (-r\sin(\theta)) \right] d\theta \\
&+ \int_0^{2\pi} \left[\left(\frac{r\cos(\theta)}{r^2} - 2r\sin(\theta) \right) (r\cos(\theta)) \right] d\theta \\
&= \int_0^{2\pi} (1 - r^2\cos^2(\theta)\sin(\theta) - 2r^2\sin(\theta)\cos(\theta)) \, d\theta \\
&= \theta + \frac{r^3}{3}\cos^2(\theta) - r^2\sin^2(\theta) \Big|_0^{2\pi} = 2\pi.
\end{aligned}$$

17. By a calculation like those of Problems 15 and 16, obtain

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = 0$$

if C does not enclose the origin, and

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \oint_K \mathbf{F} \cdot d\mathbf{R} = 0$$

if C does enclose the origin, and K is a circle about the origin entirely in the region enclosed by C.

16.3 Independence of Path and Potential Theory

1. First observe that

$$\frac{\partial}{\partial y}(y^3) = \frac{\partial}{\partial x}(3xy^2 - 4)$$

on the entire plane, so this vector function is conservative. To find a potential function, we can begin with

$$\frac{\partial \varphi}{\partial x} = y^3$$

and integrate with respect to x to get

$$\varphi(x, y) = xy^3 + k(y).$$

Then we must have

$$\frac{\partial \varphi}{\partial y} = 3xy^2 + k'(y) = 3xy^2 - 4$$

to conclude that $k'(y) = -4$, so we can choose $k(y) = -4y$. Then

$$\varphi(x, y) = xy^3 - 4y$$

is a potential function for $\mathbf{F}$.

3. $\mathbf{F}$ is conservative over the entire plane because

$$\frac{\partial}{\partial y}(16x) = \frac{\partial}{\partial x}(2 - y^2) = 0$$

for all (x, y). To find a potential function, we can begin with

$$\frac{\partial \varphi}{\partial x} = 16x$$

and integrate with respect to y to get

$$\varphi(x, y) = 8x^2 + k(y).$$

Then

$$\frac{\partial \varphi}{\partial y} = k'(y) = 2 - y^2$$

so $k(y) = 2y - y^3/3$ and

$$\varphi(x, y) = 8x^2 + 2y - \frac{1}{3}y^3$$

is a potential function.

5. First, if $(x, y) \neq (0, 0)$, then

$$\frac{\partial}{\partial y}\left(\frac{2x}{x^2 + y^2}\right) = -\frac{4xy}{(x^2 + y^2)^2} = \frac{\partial}{\partial y}\left(\frac{2y}{x^2 + y^2}\right).$$

Then $\mathbf{F}$ is conservative on the plane with the origin removed. For a potential function, we can begin with

$$\frac{\partial \varphi}{\partial x} = \frac{2x}{x^2 + y^2}$$

and integrate with respect to x to get

$$\varphi(x, y) = \ln(x^2 + y^2) + k(y).$$

Then we need
$$\frac{\partial \varphi}{\partial y} = \frac{2y}{x^2+y^2} = \frac{2y}{x^2+y^2} + k'(y).$$
We can choose $k(y) = 0$ to obtain the potential function
$$\varphi(x, y) = \ln(x^2 + y^2).$$

7. By inspection,
$$\varphi(x, y, z) = x - 2y + z$$
is a potential function for $\mathbf{F}$, for all (x, y, z).

9. We find that $\nabla \times \mathbf{F} \neq \mathbf{O}$, so this vector field is not conservative.

In Problems 11–20 we provide a potential function to use in evaluating the line integral, but do not include the details of finding this potential function.

11. By integrating, we find a potential function
$$\varphi(x, y) = x^3(y^2 - 4y).$$
Then
$$\int_C \mathbf{F} \cdot d\mathbf{R} = \varphi(2, 3) - \varphi(1, 1) = -24 - 3 = -27.$$

13. In any region not containing points of the $y-$ axis, we can use the potential function
$$\varphi(x, y) = x^2y - \ln|y|.$$
If C does not cross the $x-$ axis, then
$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{R} &= \varphi(2, 2) - \varphi(1, 3) \\ &= 8 - \ln(2) - 3 + \ln(3) = 5 + \ln(3/2).\end{aligned}$$

15. $\varphi(x, y) = x^3y^2 - 6xy^3$, so
$$\int_C \mathbf{F} \cdot d\mathbf{R} = \varphi(1, 1) - \varphi(0, 0) = -5.$$

17. $\varphi(x, y, z) = x - 3y^3z$, so
$$\int_C \mathbf{F} \cdot d\mathbf{R} = \varphi(0, 3, 5) - \varphi(1, 1, 1) = -403.$$

19. $\varphi(x, y, z) = 2x^3e^{yz}$, so
$$\int_C \mathbf{F} \cdot d\mathbf{R} = \varphi(2, 1, -1) - \varphi(0, 0, 0) = 2e^{-2}.$$

21. Let C be a smooth path of motion having position vector $\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Let $L(t)$ be the sum of the potential and kinetic energies. Then

$$\begin{aligned} L(t) &= \frac{m}{2} \parallel \mathbf{R}'(t) \parallel - \varphi((xt), y(t), z(t)) \\ &= \frac{m}{2}\mathbf{R}'(t) \cdot \mathbf{R}'(t) - \varphi(x(t), y(t), z(t)). \end{aligned}$$

Then

$$\begin{aligned} L'(t) &= \frac{m}{2}(2\mathbf{R}''(t) \cdot \mathbf{R}'(t)) - \frac{\partial \varphi}{\partial x}x'(t) - \frac{\partial \varphi}{\partial y}y'(t) - \frac{\partial \varphi}{\partial z}z'(t) \\ &= (m\mathbf{R}''(t) \cdot \mathbf{R}'(t) - \nabla\varphi \cdot \mathbf{R}'(t) \\ &= (m\mathbf{R}''(t) - \nabla\varphi) \cdot \mathbf{R}'(t). \end{aligned}$$

Now, $\nabla\varphi$ is the force acting on the particle, so by Newton's second law,

$$m\mathbf{R}'' = \nabla\varphi.$$

Therefore $L'(t) = 0$.

16.4 Surface Integrals

1. On the surface, $z = 10 - x - 4y$, so

$$d\sigma = \sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}\, dA = 3\sqrt{2}\, dA.$$

Then

$$\begin{aligned} \iint_\Sigma x\, d\sigma &= \iint_D 3\sqrt{2}x\, dA \\ &= 3\sqrt{2}\int_0^{5/2}\int_0^{10-4y} x\, dx\, dy = \frac{3\sqrt{2}}{2}\int_0^{5/2}(10-4y)^2\, dy \\ &= \frac{\sqrt{2}}{8}(10-4y)^3\Big|_0^{5/2} = 125\sqrt{2}. \end{aligned}$$

3. On Σ,

$$d\sigma = \sqrt{1 + 4x^2 + 4y^2}\, dA,$$

and D is the annulus $2 \le x^2 + y^2 \le 7$. Then, using polar coordinates,

$$\begin{aligned} \iint_\Sigma d\sigma &= \int_0^{2\pi}\int_{\sqrt{2}}^{\sqrt{7}} r\sqrt{1+4r^2}r\, dr\, d\theta \\ &= 2\pi\left[\frac{1}{12}(1+4r^2)^{3/2}\right]_{\sqrt{2}}^{\sqrt{7}} = \frac{\pi}{6}(29^{3/2} - 27). \end{aligned}$$

5. On the surface, $z^2 = x^2 + y^2$, so

$$2z\frac{\partial z}{\partial x} = 2x \text{ and } 2z\frac{\partial z}{\partial y} = 2y.$$

Then

$$\frac{\partial z}{\partial x} = \frac{x}{z} \text{ and } \frac{\partial z}{\partial y} = \frac{y}{z}.$$

Then

$$d\sigma = \sqrt{1 + \left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2}\, dA = \sqrt{2}\, dA.$$

Then

$$\begin{aligned}\iint_\Sigma z\, d\sigma &= \iint_D \sqrt{2}\sqrt{x^2+y^2}\, dA \\ &= \sqrt{2}\int_0^{\pi/2}\int_2^4 r^2\, dr\, d\theta = \frac{28\pi}{3}\sqrt{2}.\end{aligned}$$

7. On the surface, $d\sigma = \sqrt{1+4x^2}\, dA$, so

$$\begin{aligned}\int_\Sigma y\, d\sigma &= \iint_D y\sqrt{1+4x^2}\, dA \\ &= \int_0^2\int_0^3 y\sqrt{1+4x^2}\, dy\, dx = \frac{9}{2}\int_0^2 \sqrt{1+4x^2}\, dx \\ &= \frac{9}{8}\ln(4+\sqrt{17}+4\sqrt{17}).\end{aligned}$$

9. On Σ, $d\sigma = \sqrt{3}\, dA$ and $z = x - y$, so

$$\begin{aligned}\iint_\Sigma z\, d\sigma &= \iint_D \sqrt{3}(x-y)\, dA \\ &= \sqrt{3}\int_0^1\int_0^5 (x-y)\, dy\, dx = -10\sqrt{3}.\end{aligned}$$

16.5 Applications of Surface Integrals

1. The triangular shell is on the plane $6x + 2y + 3z = 6$, which is the plane through the three given points. The projection of Σ onto the $x, y-$plane is the set D of points (x, y) such that $0 \le y \le 3 - 2x$. On Σ,

$$z = 2 - \frac{2}{3}y - 2x.$$

Then

$$d\sigma = \sqrt{1 + \frac{4}{9} + 4}\, dA$$

and

$$
\begin{aligned}
m &= \iint_\Sigma (xz+1)\,d\sigma \\
&= \iint_D \left(x \left(2 - \frac{2}{3}y - 2x \right) + 1 \right)\,dA \\
&= \frac{7}{3} \int_0^1 \int_0^{3-3x} \left(x \left(2 - \frac{2}{3}y - 2x \right) + 1 \right)\,dy\,dx = \frac{49}{12}.
\end{aligned}
$$

The first coordinate of the center of mass is

$$
\begin{aligned}
\overline{x} &= \frac{12}{49} \int_\Sigma x(xz+1)\,d\sigma \\
&= \frac{12}{49}\frac{7}{3} \iint_D x \left(x \left(2 - \frac{2}{3}y - 2x \right) + 1 \right)\,dy\,dx \\
&= \frac{4}{7} \int_0^1 \int_0^{3-3x} x \left(x \left(2 - \frac{2}{3}y - 2x \right) + 1 \right)\,dx \\
&= \int_0^1 \left(-\frac{24}{7}x^2 + \frac{12}{7}x^4 + \frac{12}{7}x \right)\,dx = \frac{12}{35}.
\end{aligned}
$$

The second coordinate is

$$
\begin{aligned}
\overline{y} &= \frac{12}{49} \iint_\Sigma y(xz+1)\,d\sigma \\
&= \frac{4}{7} \int_0^1 \int_0^{3-3x} y \left(y \left(2 - \frac{2}{3}y - 2x \right) + 1 \right)\,dy\,dx \\
&= \int_0^1 \left(-\frac{24}{7}x - \frac{18}{7}x^2 + \frac{36}{7}x^3 - \frac{12}{7}x^4 + \frac{18}{7} \right)\,dx \\
&= \frac{33}{35}.
\end{aligned}
$$

And, without all the details, the third coordinate is

$$\overline{z} = \frac{12}{49} \iint_\Sigma z(xz+1)\,d\sigma = \frac{24}{35}.$$

3. On the surface,

$$d\sigma = \sqrt{1 + (x/z)^2 + (y/z)^2}\,dA = \sqrt{2}\,dA.$$

Then

$$\text{mass } = \iint_\Sigma K\,d\sigma = K\sqrt{2} \iint_D \int_0^{2\pi} \int_0^3 r\,dr\,d\theta = 9\pi K\sqrt{2}.$$

By symmetry, $\overline{x} = \overline{y} = 0$, and

$$
\begin{aligned}
\overline{z} &= \frac{1}{m} \iint_\Sigma z\,d\sigma \\
&= \frac{\sqrt{2}K}{m} \int_0^{2\pi} \int_0^3 r^2\,dr\,d\theta = \frac{18\sqrt{2}K\pi}{m} = 2.
\end{aligned}
$$

The center of mass is $(0, 0, 2)$.

5. By symmetry of the surface and the density function, $\overline{x} = \overline{y} = 0$. Further,

$$d\sigma = \sqrt{1 + 4x^2 + 4y^2}\, dA.$$

Then

$$\begin{aligned} m &= \iint_\Sigma \sqrt{1 + 4x^2 + 4y^2}\, d\sigma \\ &= \iint_D (1 - 4x^2 + 4y^2)\, dA \\ &= \int_0^{2\pi} \int_0^{\sqrt{6}} (1 + 4r^2) r\, dr\, d\theta \\ &= 2\pi(39) = 78\pi. \end{aligned}$$

Finally,

$$\begin{aligned} \overline{x} &= \frac{1}{m} \iint_\Sigma z\delta(x, y, z)\, d\sigma \\ &= \frac{1}{m} \iint_D (6 - x^2 - y^2)(1 + 4x^2 + 4y^2)\, dA \\ &= \frac{1}{m} \int_0^{2\pi} \int_0^{\sqrt{6}} (6 - r^2)(1 + 4r^2) r\, dr\, d\theta \\ &= \frac{162\pi}{m} = \frac{27}{13}. \end{aligned}$$

7. A unit normal to the plane $x + 2y + z = 8$ is

$$\mathbf{n} = \frac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} + \mathbf{k}).$$

Then

$$\mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{6}}(x + 2y - z).$$

On Σ, $z = 8 - x - 2y$, so

$$\mathbf{F} \cdot \mathbf{n} = 2x + 4y - 8.$$

Further, $d\sigma = \sqrt{1 + 4 + 1}\, dA = \sqrt{6}\, dA$, so the flux of $\mathbf{F}$ across the surface is

$$\iint_\Sigma \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_D (2x + 4y - 8)\, dA = \int_0^4 \int_0^{8-2y} (2x + 4y - 8)\, dx\, dy = \frac{128}{3}.$$

16.6 Gauss's Divergence Theorem

1. $\nabla \cdot \mathbf{F} = 1$, so

$$\iint_{\Sigma} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint\limits_{M} \nabla \cdot \mathbf{F}\, dV$$
$$= \text{ volume of } V = \frac{4}{3}\pi 4^3 = \frac{256\pi}{3}.$$

3. $\nabla \cdot \mathbf{F} = 0$, so

$$\iiint\limits_{M} \nabla \cdot \mathbf{F}\, dV = 0.$$

5. With $\nabla \cdot \mathbf{F} = 4$, compute

$$\iiint\limits_{M} \nabla \cdot \mathbf{F}\, dV = 4(\text{volume of })V = \frac{8\pi}{3}.$$

7. $\nabla \cdot \mathbf{F} = 2(x + y + z)$, so, using cylindrical coordinates, we have

$$\iiint\limits_{M} \nabla \cdot \mathbf{F}\, dV = 2\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{2}} (r\cos(\theta) + r\sin(\theta) + z)r\, dz\, dr\, d\theta.$$

Do these integrations in turn. First,

$$\int_r^{\sqrt{2}} (r^2(\cos(\theta)+\sin(\theta))+rz)\, dz = r^2(\cos(\theta)+\sin(\theta))(\sqrt{2}-r)+\frac{1}{2}r(2-r^2).$$

Next,

$$\int_0^{\sqrt{2}} \left[r^2(\cos(\theta)+\sin(\theta))(\sqrt{2}-r)+\frac{1}{2}r(2-r^2)\right] dr = \frac{1}{3}(\cos(\theta)+\sin(\theta))+\frac{1}{2},$$

and finally,

$$\int_0^{2\pi} \left(\frac{1}{3}(\cos(\theta)+\sin(\theta))+\frac{1}{2}\right) d\theta = \pi.$$

Therefore

$$\iiint\limits_{M} \nabla \cdot \mathbf{F}\, dV = 2\pi.$$

9. With the given conditions on $\mathbf{F}$, Σ and M, we have

$$\iint_{\Sigma} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \iiint\limits_{M} \nabla \cdot (\nabla \times \mathbf{F})\, dV = 0$$

because $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

10. Apply the divergence theorem to get

$$\frac{1}{3}\iint_{\Sigma} \mathbf{F}\cdot\mathbf{n}\,d\sigma = \frac{1}{3}\iiint_{M} (\nabla\cdot\mathbf{R})\,dV$$
$$= \frac{1}{3}\iiint_{M} 3\,dV = \text{ volume of } M.$$

16.7 Stokes's Theorem

1. The surface is a function of θ and φ, and (θ,φ) varies over its parameter domain $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$. This is a rectangle in the $\theta,\varphi-$ plane, with lower side L_1, upper right side L_2, top L_3 and left side L_4. For orientation, imagine (θ,φ) moves around this rectangle counterclockwise, starting along L_1 from the origin. We want to know what each side maps to on the surface.

   On L_1, the point $(\theta,0)$ moves from $(0,0)$ to $(2\pi,0)$ as θ increases from 0 to 2π. The image point

$$\Sigma(\theta,0) = (R\cos(\theta), R\sin(\theta), 0)$$

   moves from $(R,0,0)$ along the circle $x^2+y^2=R^2$ in the plane $z=0$, all the way around to end at $(R,0,0)$.

   Then the point $(2\pi,\varphi)$ moves up along L_2 as φ increases from 0 to π. Image points of points $(2\pi,\varphi)$ on L_2 are

$$\Sigma(2\pi,\varphi) = (R\cos(\varphi), 0, R\sin(\varphi))$$

   which is a half-circle $x^2+z^2=R^2$ in the $y=0$ plane, starting at $(R,0,0)$ and ending at $(-R,0,0)$.

   From $(2\pi,\pi)$, (θ,φ) now moves left along L_3. The points are (θ,π), but θ varies from 2π to 0 to maintain counterclockwise orientation on the rectangle. The image points of L_3 are

$$\Sigma(\theta,\pi) = (-R\cos(\theta), -R\sin(\theta), 0)$$

   asθ varies from2π to 0. The image of L_3 on the surface consists of the points

$$\Sigma(\theta,\pi) = (-R\cos(\theta), -R\sin(\theta), 0),$$

   and this point moves along the half-circle

$$x^2+y^2=R^2$$

   from $(-R,0,0)$ to $(-R,0,0)$ in the $z=0$ plane.

   Finally, on L_4, $\theta=0$ and φ varies from π to 0. Image points are

$$\Sigma(0,\varphi) = (R\cos(\varphi), 0, R\sin(\varphi))$$

from $(-R, 0, 0)$ to $(R, 0, 0)$.

Now trace out the image point on the surface as (θ, φ) moves over all four sides of the rectangle. This curve on the graph of the surface is the boundary of Σ.

In Problems 3–8, one side of Stokes's theorem is computed in detail, with the choice being determined by which side appears to be the easiest computation.

3. The boundary curve C of the surface is the top of the parabolic bowl. This is the circle of radius 3 about $(0, 0, 9)$. Parametrize C by

$$x = 3\cos(t), y = 3\sin(t), z = 9 \text{ for } 0 \le t \le 2\pi.$$

On C,

$$\mathbf{F}(t) = 9\cos(t)\sin(t)\mathbf{i} + 27\sin(t)\mathbf{j} + 27\cos(t)\mathbf{k}.$$

Further,

$$d\mathbf{R} = (-3\sin(t)\mathbf{i} + 3\cos(t)\mathbf{j})\, dt.$$

Then

$$\mathbf{F} \cdot d\mathbf{R} = (-27\cos(t)\sin^2(t) + 81\cos(t)\sin(t))\, dt.$$

A routine integration gives

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \int_0^{2\pi} (-27\cos(r)\sin^2(t) + 81\cos(t)\sin(t))\, dt = 0.$$

Evaluation of $\iint_\Sigma (\nabla \times \mathbf{F})\, d\sigma$ involves considerably more labor.

5. The boundary curve of Σ is the circle $x^2 + y^2 = 6$ in the $x, y-$ plane. Parametrize C by

$$x = \sqrt{6}\cos(t), y = \sqrt{6}\sin(t), z = 0 \text{ for } 0 \le t \le 2\pi.$$

Then

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \int_0^{2\pi} 6\cos^2(t) 6\sin^2(t)\, dt = 0.$$

7. The circulation is $\oint_C \mathbf{F} \cdot d\mathbf{R}$. Take Σ to be the disk

$$0 \le x^2 + y^2 \le 1$$

with boundary C parametrized by

$$x = \cos(t), y = \sin(t), z = 0 \text{ for } 0 \le t \le 2\pi.$$

The proper unit normal to Σ (a disk in the $x, y-$ plane) is $\mathbf{n} = \mathbf{k}$. Now,

$$\nabla \times \mathbf{F} = -az\mathbf{j} + (2xy + 1)\mathbf{k}.$$

Then

$$(\nabla \times \mathbf{F}) \cdot \mathbf{n} = 2xy + 1.$$

Further, $d\sigma = dA$, so

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{R} &= \iint_\Sigma (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma \\ &= \iint_D (2xy + 1)\, dA = \int_0^{2\pi} \int_0^1 (2r^3 \cos(\theta) \sin(\theta) + r)\, dr\, d\theta = \pi.\end{aligned}$$

Chapter 17

Fourier Series

17.1 Fourier Series on $[-L, L]$

1. The Fourier coefficients are

$$a_0 = \frac{1}{3}\int_{-3}^{3} 4\,dx = 8,$$

$$a_n = \frac{1}{3}\int_{-3}^{3} 4\cos(n\pi\xi/3)\,d\xi = 0,$$

and

$$b_n = \frac{1}{3}\int_{-3}^{3} 4\sin(n\pi\xi/3)\,d\xi = 0.$$

The Fourier series of 4 on $[-3, 3]$ is just

$$\frac{1}{2}a_0$$

or 4, as we might expect. This converges to 4 on $[-3, 3]$.

3. Because $\cosh(\pi x)$ is an even function on $[-1, 1]$, each $b_n = 0$. Compute

$$a_0 = \int_{-1}^{1} \cosh(\pi\xi)\,d\xi = \frac{2}{\pi}\sinh(\pi)$$

and, for $n = 1, 2, \cdots$,

$$a_n = \int_{-1}^{1} \cosh(\pi\xi)\cos(n\pi\xi)\,d\xi = \frac{2\sinh(\pi)}{\pi}\frac{(-1)^n}{n^2+1}.$$

The Fourier series is

$$\frac{1}{\pi}\sinh(\pi) + \sum_{n=1}^{\infty} \frac{2\sinh(\pi)}{\pi}\frac{(-1)^n}{n^2+1}\cos(n\pi x).$$

This converges to $\cosh(\pi x)$ for $-1 \le x \le 1$.

For Problems 4–10, we give just the Fourier series and analyze its convergence.

5. The series of $f(x)$ on $[-\pi, \pi]$ is

$$\frac{16}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)x),$$

converging to

$$\begin{cases} -4 & \text{for } -\pi < x < 0, \\ 4 & \text{for } 0 < x < \pi, \\ 0 & \text{for } x = \pi \text{ and for } x = -\pi. \end{cases}$$

7. The Fourier series of $f(x)$ on $[-2, 2]$ is

$$\frac{13}{3} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{16}{(n\pi)^2}\cos(n\pi x/2) + \frac{4}{n\pi}\sin(n\pi x/2)\right].$$

This converges to

$$\begin{cases} f(x) & \text{for } -2 < x < 2, \\ 7 & \text{for } x = 2 \text{ and for } x = -2. \end{cases}$$

Convergence at the endpoints is determined by

$$\frac{1}{2}(f(-2+) + f(2-)) = \frac{1}{2}(9+5) = 7.$$

9. The Fourier expansion of $f(x)$ on $[-\pi, \pi]$ is

$$\frac{3}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)x)$$

converging to

$$\begin{cases} 1 & \text{for } -\pi < x < 0, \\ 2 & \text{for } 0 < x < \pi, \\ 3/2 & \text{at } x = 0,\ x = \pi \text{ and at } x = -\pi. \end{cases}$$

11. The Fourier series is

$$\frac{1}{3}\sin(3) + 6\sin(3)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2\pi^2 - 9}\cos(n\pi x/3),$$

converging to $\cos(x)$ on $[-3, 3]$.

13. The Fourier series converges to

$$\begin{cases} 3/2 & \text{for } x = \pm 3, \\ 2x & \text{for } -3 < x < -2, \\ -2 & \text{for } x = -2, \\ 0 & \text{for } -2 < x < 1, \\ 1/2 & \text{for } x = 1, \\ x^2 & \text{for } 1 < x < 3. \end{cases}$$

15.

$$\begin{cases} (2+\pi^2)/2 & \text{for } x = \pm\pi, \\ x^2 & \text{for } -\pi < x < 0, \\ 1 & \text{for } x = 0, \\ 2 & \text{for } 0 < x < \pi \end{cases}$$

17.

$$\begin{cases} -1 & \text{for } -4 < x < 0, \\ 0 & \text{for } x = \pm 4 \text{ and for } x = 0, \\ 1 & \text{for } 0 < x < 4 \end{cases}$$

19.

$$\begin{cases} -4 & \text{for } x = \pm 4, \\ 3/2 & \text{for } x = -2, \\ 5/2 & \text{for } x = 2, \\ f(x) & \text{for all other } x \text{ in } [-4, 4] \end{cases}$$

17.2 Sine and Cosine Series

1. The cosine series is just 4, a single term. The sine series is

$$\frac{16}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)\pi x/3),$$

converging to 0 for $x = 0$ or $x = 3$ and to 4 for $0 < x < 3$.

3. The cosine series is

$$\frac{1}{2}\cos(x) + \sum_{n=1,n\neq 2}^{\infty}\frac{2n\sin(n\pi/2)}{\pi(n^2-4)}\cos(nx/2)$$

converging to

$$\begin{cases} 0 & \text{for } 0 < x < \pi, \\ -1/2 & \text{for } x = \pi, \\ \cos(x) & \text{for } \pi < x < 2\pi, \\ 1 & \text{for } x = 2\pi. \end{cases}$$

The sine series is

$$-\frac{2}{3\pi}\sin(x/2)-\sum_{n=3}^{\infty}\frac{2n}{(n^2-4)}((-1)^n+\cos(n\pi/2))\sin(nx/2),$$

converging to

$$\begin{cases} 0 & \text{for } 0<x<\pi, \\ -1/2 & \text{for } x=\pi, \\ \cos(x) & \text{for } \pi<x<2\pi, \\ 0 & \text{for } x=2\pi. \end{cases}$$

5. The cosine series is

$$\frac{4}{3}+\frac{16}{\pi^2}\frac{(-1)^n}{n^2}\cos(n\pi x/2),$$

converging to x^2 for $0\le x\le 2$. The sine expansion is

$$-\frac{8}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n}+\frac{2(1-(-1)^n)}{n^3\pi^2}\right]\sin(n\pi x/2),$$

converging to x^2 for $0\le x<2$ and to 0 at $x=2$.

7. The cosine expansion is

$$\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\left[-6(1+(-1)^n)+12\cos(2n\pi/3)+4n\pi\sin(2n\pi/3)\right]\cos(n\pi x/3),$$

converging to

$$\begin{cases} x & \text{for } 0\le x<2, \\ 1 & \text{for } x=2, \\ 2-x & \text{for } 2<x\le 3. \end{cases}$$

The sine series is

$$\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\left[12\sin(2n\pi/3)-4n\pi\cos(2n\pi/3)+12n\pi(-1)^n\right]\sin(n\pi x/3),$$

converging to

$$\begin{cases} x & \text{for } 0\le x<2, \\ 1 & \text{for } x=1, \\ 2-x & \text{for } 2<x<3, \\ 0 & \text{for } x=3. \end{cases}$$

9. The cosine expansion is

$$\frac{5}{6} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{1}{n^2} \cos(n\pi/4) - \frac{4}{n^3\pi} \sin(n\pi/4) \right] \cos(n\pi x/4),$$

converging to x^2 for $0 \le x < 1$ and to 1 for $1 < x \le 4$.

The sine series is

$$\sum_{n=1}^{\infty} \left[\frac{16}{n^2\pi^2} \sin(n\pi/4) + \frac{61}{n^3\pi^3} (\cos(n\pi/4) - 1) - \frac{2}{(-1)^n} n\pi \right] \sin(n\pi x/4),$$

converging to x^2 for $0 \le x < 1$, to 1 if $1 < x < 4$, and to 0 at $x = 4$.

11. The Fourier cosine expansion of $\sin(x)$ on $[0, \pi]$ is

$$\frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cos(2nx).$$

This converges to $\sin(x)$ for $0 \le x \le \pi$. Put $x = \pi/2$ into this series to obtain

$$\cos(\pi/2) = 0 = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cos(n\pi).$$

Upon putting $\cos(n\pi) = (-1)^n$ we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} = \frac{\pi}{4} \left(\frac{2}{\pi} - 1 \right) = \frac{1}{2} - \frac{\pi}{4}.$$

13. Use an argument similar to that made for Problem 12, except now the Fourier coefficients of $G_e(x)$ satisfy

$$\begin{aligned} a_n &= \frac{-L}{L} \int_{-L}^{L} G_e(x) \cos(n\pi x/L)\, dx \\ &= \frac{2}{L} \int_0^L g(x) \cos(n\pi x/L)\, dx \end{aligned}$$

and

$$b_n = \frac{1}{L} \int_{-L}^{L} G_e(x) \sin(n\pi x/L)\, dx = 0$$

because $G_e(x)\cos(n\pi x/L)$ is even and $G_e(x)\sin(n\pi x/L)$ is odd, and $G_e(x) = g(x)$ for $0 \le x \le L$.

17.3 Integration and Differentiation of Fourier Series

1. The Fourier expansion of $f(x)$ on $[-\pi, \pi]$ is

$$\frac{1}{4}\pi + \sum_{n=1}^{\infty} \left(\frac{(-1)^n - 1}{n^2\pi} \cos(nx) + \frac{(-1)^n}{n} \sin(nx) \right).$$

Because f is continuous and piecewise smooth on $[-\pi, \pi]$, this series converges to $f(x)$ for $-\pi < x < \pi$.

$f(x)$ satisfies the conditions of the theorem on term by term integration, so this series can be integrated term by term to obtain

$$\begin{aligned}\int_{-\pi}^{x} f(\xi)\, d\xi &= \frac{\pi}{4}(x+\pi) \\ &\quad + \sum_{n=1}^{\infty} \left(\frac{1}{n^3\pi}((-1)^n - 1)\sin(nx) + \frac{(-1)^n}{n^2}\cos(nx) - \frac{1}{n^2} \right).\end{aligned}$$

3. The Fourier expansion of $f(x)$ on $[-\pi, \pi]$ is

$$1 - \frac{1}{2}\cos(x) - 2\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 - 1}\cos(nx),$$

converging to $x\sin(x)$ for $-\pi \le x \le \pi$. The function is continuous on $[-\pi, \pi]$, and $f'(x)$ is piecewise continuous. Further, $f(-\pi) = f(\pi)$ and $f''(x)$ exists on $(-\pi, \pi)$. We can differentiate the Fourier series term by term to obtain, for $-\pi < x < \pi$,

$$\begin{aligned}f'(x) &= \sin(x) + x\cos(x) \\ &= \frac{1}{22}\cos(x) + 2\sum_{n=2}^{\infty} \frac{n(-1)^n}{n^2 - 1}\cos(nx).\end{aligned}$$

5. Let the Fourier coefficients of f on $[-L, L]$be a_n and b_n, as usual. Let the Fourier coefficients of $f'(x)$ be A_n, B_n. Notice that

$$A_0 = \frac{2}{L}\int_{-L}^{L} f'(\xi)\, d\xi = f(L) - f(_L) = 0$$

because $f(L) = f(-L)$.

Now we will develop some inequalities aimed at showing uniform convergence of the Fourier series of $f(x)$ on $[-L, L]$.

Begin with an integration by parts:

$$\begin{aligned}A_n &= \frac{1}{L}\int_{-L}^{L} f'(\xi)\cos(n\pi\xi/L)\, d\xi \\ &= \frac{1}{L}\left[f(x)\cos(n\pi x/L)\right]_{-L}^{L} + \frac{n}{\pi}\frac{1}{L}\int_{-L}^{L} f(\xi)\sin(n\pi\xi/L)\, d\xi.\end{aligned}$$

Now,

$$f(L)\cos(n\pi) - f(-L)\cos(-n\pi) = 0$$

for integer n, again because $f(L) = f(_L)$. Then

$$A_n = \frac{n\pi}{L}\frac{1}{L}\int_{-L}^{L} f(\xi)\cos(n\pi\xi/L)\,d\xi = \frac{n\pi}{L}a_n$$

for $n = 1, 2, \cdots$. A similar integration by parts gives us

$$B_n = -\frac{n\pi}{L}a_n.$$

Observe that

$$0 \leq \left(|A_n| - \frac{1}{n}\right)^2 = A_n^2 - \frac{2}{n}|A_n| + \frac{1}{n^2}$$

and, similarly,

$$0 \leq B_n^2 - \frac{2}{n}|B_n| + \frac{1}{n^2}.$$

Add these two inequalities to get

$$\frac{2}{n}(|A_n| + |B_n|) \leq A_n^2 + B_n^2 + \frac{2}{n^2}.$$

Multiply this by $1/2$ to obtain

$$\frac{1}{n}(|A_n| + |B_n|) \leq \frac{1}{2}(A_n^2 + B_n^2) + \frac{1}{n^2}.$$

On the left, insert

$$|A_n| = \frac{n\pi|a_n|}{L} \text{ and } |B_n| = \frac{n\pi|a_n|}{L}$$

to obtain

$$|a_n| + |b_n| \leq \frac{L}{2\pi}(A_n^2 + B_n^2) + \frac{L}{\pi}\frac{1}{n^2}.$$

Now,

$$\sum_{n=1}^{\infty} A_n^2 \text{ and } \sum_{n+1}^{\infty} B_n^2$$

both converge, by Bessel's inequality. Therefore, by the comparison test for nonnegative series, we conclude that

$$\sum_{n=1}^{\infty}(|a_n| + |b_n|)$$

converges. Finally, observe that, on $[-L, L]$,

$$|a_n\cos(n\pi x/L)| + b_n\sin(n\pi x/L)| \leq |a_n| + |b_n|.$$

By what is often known as the Weierstrass M- test (in this case with

$$M_n = |a_n| + |b_n|$$

the Fourier series of $f(x)$ on $[-L, L]$ converges uniformly on this interval.

17.4 Properties of Fourier Coefficients

1. The argument is like that for sine series, but is notationally a little messier because of the additional constant term in the cosine expansion. Let

$$S_N = \frac{1}{2}A_0 + \sum_{n=1}^{N} A_n \cos(n\pi x/L).$$

Now

$$\begin{aligned}
0 &\le \int_0^L (g(x) - S_N(x))^2 \, dx \\
&= \int_0^L \left((g(x))^2 \, dx - 2g(x)S_N(x) + S_N^2(x)\right) \, dx \\
&= \int_0^L (g(x))^2 \, dx - 2\int_0^L g(x) \left(\frac{1}{2}A_0 + \sum_{n=1}^{N} A_n \cos(n\pi x/L)\right) \, dx \\
&+ \int_0^L \left(\frac{1}{2}A_0 + \sum_{n=1}^{N} A_n \cos(n\pi x/2)\right)\left(\frac{1}{2}A_0 + \sum_{n=1}^{N} A_n \cos(n\pi x/L)\right) \, dx \\
&= \int_0^L (g(x))^2 \, dx - \int_0^L A_0 g(x) \, dx \\
&- 2\sum_{n=1}^{N} \int_0^L g(x) A_n \cos(n\pi x/L) \, dx \\
&+ \int_0^L \left(\frac{1}{4}A_0^2 + \sum_{n=1}^{N}\sum_{m=1}^{N} A_n A_m \cos(n\pi x/L)\cos(m\pi x/L) + \sum_{n=1}^{N} A_0 A_n \cos(n\pi x/L)\right) \, dx \\
&= \int_0^L (g(x))^2 \, dx - \frac{L}{2}A_0^2 - L\sum_{n=1}^{N} A_n^2 \\
&+ \frac{L}{4}A_0^2 + \sum_{n=1}^{N} \frac{L}{2}A_n^2 \\
&= \int_0^L (g(x))^2 - \frac{L}{4}A_0^2 - \frac{L}{2}\sum_{n=1}^{N} A_n^2.
\end{aligned}$$

Here we have used the fact that

$$\int_0^L \cos(n\pi x/L) \, dx = 0$$

and that

$$\int_0^L \cos(n\pi x/L)\cos(m\pi x/L) \, dx = \begin{cases} L/2 & \text{if } n = m, \\ 0 & \text{if } n \neq m, \end{cases}$$

Upon rearranging terms in the first and last lines (which are connected by an inequality), we have

$$\frac{1}{2}A_0^2 + \sum_{n=1}^{N} A_n^2 \le \frac{2}{L}\int_0^L (g(x))^2\,dx.$$

Problems 3 and 4 are obtained by adapting the argument of the text to the notation cosine expansions on $[0, L]$ and Fourier series on $[-L, L]$, similar to the solution of Problem 1.

17.5 Phase Angle Form

1.

$$\begin{aligned}(\alpha f + \beta g)(x+p) &= \alpha f(x+p) + \beta g(x+p)\\ &= \alpha f(x) + \beta g(x) = (\alpha f + \beta g)(x).\end{aligned}$$

3.

$$\begin{aligned}f'(x+p) &= \lim_{h\to 0}\frac{f(x+p+h) - f(x+p)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = f'(x).\end{aligned}$$

5. The Fourier series is

$$\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)\pi x),$$

with phase angle form

$$1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\cos\left((2n-1)\pi x - \frac{\pi}{2}\right).$$

Points of the amplitude spectrum are

$$(0, 1), (n\pi, 1/((2n-1)\pi)) \text{ for } n = 1, 2, \cdots.$$

7. The Fourier series is

$$\frac{19}{8} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}\alpha_n\cos(n\pi x/2) + \beta_n\sin(n\pi x/2),$$

where

$$\alpha_n = n\pi\sin(3n\pi/2) + \cos(3n\pi/2)$$

and

$$\beta_n = \sin(3n\pi/2) - \frac{n\pi}{2} - n\pi\cos(3n\pi/2).$$

The phase angle form is

$$\frac{19}{8} + \frac{1}{\pi^2}\sum_{n=1}^{\infty} d_n \cos\left(\frac{n\pi x}{2} + \delta_N\right),$$

where

$$d_n = \sqrt{8 + 5n^2\pi^2 - 12n\pi\sin(3n\pi/2) + 4(n^2\pi^2 - 2)\cos(3n\pi/2)}$$

and

$$\delta_n = \arctan\left(\frac{n\pi/2 + n\pi\cos(3n\pi/2) - \sin(3n\pi/2)}{n\pi\sin(3n\pi/2) + \cos(3n\pi/2) - 1}\right).$$

9. We can write

$$f(x) = \begin{cases} x & \text{for } 0 \le x < 1, \\ x - 2 & \text{for } 1 < x < 2 \end{cases}$$

and $f(x+2)f(x)$, so f has period 2. The Fourier series is

$$\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(n\pi x).$$

The phase angle form is

$$\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\cos\left(n\pi x + (-1)^{n+1}\frac{\pi}{2}\right).$$

11. We can write

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x < 1, \\ 2 & \text{for } 1 < x < 3, \\ 1 & \text{for } 3 < x < 4 \end{cases}$$

with $f(x+4) = f(x)$. The Fourier series is

$$\frac{3}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}\cos((2n-1)\pi x/2).$$

The phase angle form is

$$\frac{3}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \cos\left((2n-1)\frac{\pi x}{2} + \frac{\pi}{2}(1 - (-1)^n)\right).$$

17.6 Complex Fourier Series

1. Compute

$$d_0 = \frac{1}{3}\int_0^3 2t\,dt = 3$$

and

$$d_n = \frac{1}{3}\int_0^3 2xe^{2n\pi ix/3}\,dx = \frac{3}{n\pi}i.$$

The complex Fourier series of $f(x)$ is

$$3 + \frac{3i}{\pi}\sum_{n=-\infty,n\neq 0}^{\infty} \frac{1/n}{e}^{2n\pi ix/3}$$

converging to

$$\begin{cases} 3 & \text{for } x = 0 \text{ or } x = 3, \\ 2x & \text{for } 0 < x < 3. \end{cases}$$

Points of the frequency spectrum are

$$(0,3), \left(\frac{2n\pi}{3}, \frac{3}{n\pi}\right),$$

in which n is a nonzero integer.

3. The complex Fourier series of the function is

$$\frac{3}{4} - \frac{1}{2\pi}\sum_{n=-\infty,n\neq 0}^{\infty} \frac{1}{n}(\sin(n\pi/2) + (\cos(n\pi/2) - 1)i)e^{n\pi ix/2},$$

converging to

$$\begin{cases} 1/2 & \text{for } x = 0, 1 \text{ or } 4, \\ 0 & \text{for } 0 < x < 1, \\ 1 & \text{for } 1 < x < 4. \end{cases}$$

Points of the frequency spectrum are

$$(0,3/4), \left(\frac{n\pi}{2}, \frac{1}{2n\pi}\sqrt{\sin^2(n\pi/2) + (\cos(n\pi/2) - 1)^2}\right).$$

5. The complex Fourier series is

$$\frac{1}{2} + \frac{3i}{\pi}\sum_{n=-\infty,n\neq 0} e^{(n-1)\pi ix/2},$$

converging to

$$\begin{cases} 1/2 & \text{for } x = 0, 2, 4, \\ -1 & \text{for } 0 < x < 2, \\ 2 & \text{for } 2 < x < 4. \end{cases}$$

Points of the frequency spectrum are

$$(0, 1/2), \left(\frac{n\pi}{2}, \frac{3}{(2n-1)\pi}\right).$$

7. The complex Fourier series is

$$\frac{1}{2} - \frac{2}{\pi^2} \sum_{n=-\infty, n\neq 0}^{\infty} \frac{1}{(2n-1)^2} e^{(2n-1)\pi i x},$$

converging to $f(x)$ for $0 \leq x \leq 2$. Points of the frequency spectrum are

$$(0, 1/2), \left(n\pi, \frac{2}{\pi^2} \frac{1}{(2n-1)^2}\right).$$

17.7 Filtering of Signals

1. The complex Fourier coefficients of f are $d_0 = 0$ and, for $n \neq 0$,

$$d_n = \frac{1}{4} \left[\int_{-2}^{0} -e^{-n\pi i t/2}\, dt + \int_{0}^{2} e^{-n\pi i t/2}\, dt\right] \qquad = \frac{i}{n\pi}((-1)^n - 1).$$

The complex Fourier series is

$$\sum_{n=-\infty, n\neq 0}^{\infty} \frac{i}{n\pi}((-1)^n - 1) e^{n\pi i t/2}.$$

After some routine calculation using Euler's formula, we obtain the series

$$\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin((2n-1)\pi t/2).$$

The Nth partial sum is

$$S_N(t) = \frac{4}{\pi} \sum_{n=1}^{N} \frac{1}{2n-1} \sin((2n-1)\pi t/2).$$

The Nth Cesáro sum is formed by inserting factors $1 - |n|/N$:

$$\sigma_N(t) = \frac{4}{\pi} \sum_{n=1}^{N} \left(1 - \frac{2n-1}{N}\right) \frac{1}{2n-1} \sin((2n-1)\pi t/2).$$

3. We obtain

$$S_N(t) = \sum_{n=1}^{N} \frac{2}{n\pi} (\cos(n\pi/2) - (-1)^n) \sin(n\pi t)$$

and

$$\sigma_N(t) = \sum_{n=1}^{N} \left(1 - \frac{n}{N}\right) \frac{2}{n\pi} (\cos(n\pi/2) - (-1)^n) \sin(n\pi t).$$

5. We find that

$$S_N(t) = \frac{17}{4} + \sum_{n=1}^{N} \left[\frac{1-(-1)^n}{n^2\pi^2} \cos(n\pi t) + \frac{5-6(-1)^n}{n\pi} \sin(n\pi t) \right]$$

and

$$\sigma_N(t) = \frac{17}{4} + \sum_{n=1}^{N} \left(1 - \frac{n}{N}\right) \left[\frac{1-(-1)^n}{n^2\pi^2} \cos(n\pi t) + \frac{5-6(-1)^n}{n\pi} \sin(n\pi t) \right]$$

7. The partial sums are

$$S_N(t) = 1 + \sum_{n=1}^{N} \frac{2}{n\pi}(1-3(-1)^n) \sin(n\pi t/2),$$

$$\sigma_N(t) = 1 + \sum_{n=1}^{N} \frac{2}{n\pi} \left(1 - \frac{n}{N}\right)(1-3(-1)^n) \sin(n\pi t/2),$$

$$H_N(t) = 1 + \frac{2}{n\pi}(0.54 + 0.46\cos(n\pi/N))(1-3(-1)^n) \sin(n\pi t/2),$$

$$G_N(t) = 1 + \sum_{n=1}^{N} \frac{2}{n\pi} e^{-n^2\pi^2/N^2}(1-3(-1)^n) \sin(n\pi t/2).$$

Chapter 18

Fourier Transforms

18.1 The Fourier Transform

1.
$$\widehat{f}(\omega) = \int_{-1}^{0} e^{-i\omega x}\,dx + \int_{0}^{1} e^{-i\omega x}\,dx = \frac{2i}{\omega}(\cos(\omega) - 1).$$

The amplitude spectrum is the graph of

$$|\widehat{f}(\omega)| = \left|\frac{2}{\omega}(\cos(\omega) - 1)\right|.$$

3. Write
$$f(x) = 5[H(x + 4 - 7) - H(x + 4 + 7)]$$
to obtain
$$\widehat{f}(\omega) = 5e^{-7i\omega}\left(\frac{2\sin(4\omega)}{\omega}\right) = \frac{10}{\omega}e^{-7i\omega}\sin(4\omega).$$

The amplitude spectrum is a graph of

$$|\widehat{f}(\omega)| = \left|\frac{10}{\omega}\sin(4\omega)\right|.$$

5.
$$\begin{aligned}\widehat{f}(\omega) &= \int_{k}^{\infty} e^{-x/4}e^{-i\omega x}\,dx \\ &= \left.\frac{e^{-(i\omega+1/4)x}}{-(i\omega + 1/4)}\right|_{k}^{\infty} = \frac{4e^{-(i\omega+1/4)k}}{1 + 4i\omega}.\end{aligned}$$

The amplitude spectrum is a graph of

$$|\widehat{f}(\omega)| = \frac{4e^{-k/4}}{\sqrt{1 + 16\omega^2}}.$$

7.
$$\widehat{f}(\omega) = \pi e^{-|\omega|}.$$

The amplitude spectrum is a graph of this function, which is nonnegative and hence equals its own magnitude.

9.
$$\widehat{f}(\omega) = \frac{24}{16+\omega^2} e^{2i\omega}.$$

The amplitude spectrum is a graph of

$$|\widehat{f}(\omega)| = \frac{24}{16+\omega^2}.$$

11.
$$f(x) = 18\sqrt{\frac{2}{\pi}} e^{-4ix} e^{-8x^2}$$

13. Write
$$\widehat{f}(\omega) = \frac{e^{2(\omega-3)i}}{5+(\omega-3)i}$$

to obtain

$$\begin{aligned} f(x) &= e^{3ix} \widehat{f}^{-1}\left[\frac{e^{2i\omega}}{5+i\omega}\right] \\ &= e^{3ix} H(x+2) e^{-5(x+2)} = H(x+2) e^{-(10+(5-3i)x)}. \end{aligned}$$

15. Write
$$\widehat{f}(\omega) = \frac{1+i\omega}{(3+i\omega)(2+i\omega)} = \frac{2}{3+i\omega} - \frac{2}{2+i\omega}.$$

Then
$$f(x) = (2e^{-3x} - e^{-2x})H(x).$$

17.
$$\begin{aligned} \widehat{f}^{-1}\left(\frac{1}{(1+i\omega)^2}\right) &= H(x)e^{-x} * H(x)e^{-x} \\ &= \int_{-\infty}^{\infty} H(\xi)e^{-\xi} H(x-\xi) e^{-(x-\xi)}\, d\xi \\ &= H(x)e^{-x} \int_0^x d\xi = H(x)xe^{-x}. \end{aligned}$$

19. Compute
$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)\overline{\widehat{f}(\omega)}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f}(\omega)|^2\, d\omega.$$

21. Begin with

$$\mathcal{F}\left[\frac{1}{2}(H(x+3)-H(x-3))\right](\omega)=\frac{1}{2}\int_{-3}^{3}e^{-i\omega x}\,dx$$
$$=\frac{e^{3i\omega}-e^{-3i\omega}}{2i\omega}.$$

Use the symmetry property of the transform to get

$$\mathcal{F}\left[\frac{\sin(3x)}{x}\right](\omega)=\pi[H(-\omega+3)-H(-\omega-3)]$$
$$=\pi[H(\omega+3)-H(\omega-3)].$$

Now use Parseval's identity to write

$$\int_{-\infty}^{\infty}\left(\frac{\sin(3x)}{x}\right)^2dx=\frac{1}{2\pi}\int_{-3}^{3}\pi^2\,d\omega=3\pi.$$

23.

$$\widehat{f}_{\text{win}}(\omega)=\int_{-5}^{5}x^2e^{-i\omega x}\,dx$$
$$=\frac{2}{\omega^3}(25\omega^2\sin(5\omega)+10\omega\cos(5\omega)-2\sin(5\omega)).$$

Because $w(x)=1$ and the support of g is $[-5,5]$, then $t_C=0$. For the RMS bandwidth of the window function, we have

$$w_{\text{RMS}}=2\left(\frac{\int_{-5}^{5}x^2\,dx}{\int_{-5}^{5}\,dx}\right)^{1/2}=\frac{10}{\sqrt{3}}.$$

25. Compute

$$\widehat{f}_{\text{win}}(\omega)=\int_{0}^{1}e^{-x}e^{-i\omega x}\,dx=\frac{1}{1+i\omega}(1-e^{-4(1+i\omega)})$$
$$=\frac{1}{1+\omega^2}(1-e^{-4}(\cos(4\omega)-i\sin(4\omega))(1-i\omega)$$
$$=\frac{1-e^{-4}\cos(4\omega)+e^{-4}\sin(4\omega)}{1+\omega^2}$$
$$+i\left[\frac{e^{-4}\sin(4\omega)+(e^{-4}\cos(4\omega)-1)\omega}{1+\omega^2}\right].$$

We also have

$$t_C=\frac{\int_0^4 x\,dx}{\int_0^4\,dx}=2$$

and

$$w_{\text{RMS}}=2\left(\frac{\int_0^4(x-2)^2\,dx}{\int_0^4\,dx}\right)^{1/2}=\frac{4}{\sqrt{3}}.$$

27.

$$\begin{aligned}\widehat{f}_{\text{win}}(\omega) &= \int_{-2}^{2}(x+2)^2 e^{-i\omega x}\,dx \\ &= \frac{4}{\omega^3}((4\omega^2-1)\sin(2\omega)+2\omega\cos(2\omega)) \\ &+ \frac{8i}{\omega^2}(2\omega\cos(2\omega)-\sin(2\omega)).\end{aligned}$$

With $w(x) = 1$ and support $[-2, 2]$, we have $t_C = 0$. Finally,

$$w_{\text{RMS}} = 2\left(\frac{\int_{-2}^{2} x^2\,dx}{\int_{-2}^{2}\,dx}\right)^{1/2} = \frac{4}{\sqrt{3}}.$$

18.2 Fourier Sine and Cosine Transforms

In these problems the integrations are straightforward and details are omitted.

1.

$$\begin{aligned}\widehat{f}_C(\omega) &= \int_0^\infty e^{-x}\cos(\omega x)\,dx = \frac{1}{1+\omega^2}, \\ \widehat{f}_S(\omega) &= \int_0^\infty e^{-x}\sin(\omega x)\,dx = \frac{\omega}{1+\omega^2}\end{aligned}$$

3.

$$\begin{aligned}&\widehat{f}_C(\omega) = \frac{1}{2}\left[\frac{\sin(K(\omega+1))}{\omega+1} + \frac{\sin(K(\omega-1))}{\omega-1}\right] \text{ for } \omega \neq \pm 1, \\ &\widehat{f}_C(1) = \widehat{f}_C(-1) = \frac{K}{2} + \frac{1}{2}\sin(2K)\end{aligned}$$

$$\begin{aligned}&\widehat{f}_S(\omega) = \frac{\omega}{\omega^2-1} - \frac{1}{2}\left[\frac{\cos((\omega+1)K)}{\omega+1} + \frac{\cos((\omega-1)K)}{\omega-1}\right] \text{ for } \omega \neq \pm 1, \\ &\widehat{f}_S(1) = \frac{1}{4}(1-\cos(2K)), \widehat{f}_S(-1) = -\frac{1}{4}(1-\cos(2K))\end{aligned}$$

5.

$$\begin{aligned}\widehat{f}_C(\omega) &= \frac{1}{2}\left[\frac{1}{1+(\omega+1)^2} + \frac{1}{1+(\omega-1)^2}\right], \\ \widehat{f}_S(\omega) &= \frac{1}{2}\left[\frac{\omega+1}{1+(\omega+1)^2} + \frac{\omega-1}{1+(\omega-1)^2}\right]\end{aligned}$$

7. Suppose, for each positive number L, $f^{(4)}(x)$ is piecewise continuous on $[0, L]$, $f^{(3)}(x)$ is continuous, and, as $x \to \infty$, $f^{(j)}(x) \to 0$ for $j = 1, 2, 3$. Then we can integrate by parts four times to obtain

$$\mathcal{F}_S[f^{(4)}(x)](\omega) = \int_0^\infty f^{(4)}(x)\sin(\omega x)\,dx$$
$$= \Big[f^{(3)}(x)\sin(\omega x) - \omega f''(x)\cos(\omega x) - \omega^2 f'(x)\sin(\omega x) + \omega^3 f(x)\cos(\omega x)\Big]_0^\infty$$
$$+ \omega^4 \int_0^\infty f(x)\sin(\omega x)\,dx$$
$$= \omega^4 \mathcal{F}_S(\omega) - \omega^3 f(0) + \omega f''(0).$$

Chapter 19

Complex Numbers and Functions

19.1 Geometry and Arithmetic of Complex Numbers

1.
$$(3-4i)(6+2i) = (18+8) + (-24+6)i = 26 - 18i$$

3.
$$\frac{2+i}{4-7i} = \frac{2+i}{4-7i}\frac{4+7i}{4+7i} = \frac{1+18i}{65}$$

5.
$$(17-6i)(\overline{-3-12i}) = (17-6i)(-3+12i) = 4 + 228i$$

7.
$$i^3 - 4i^2 + 2 = -i + 4 + 2 = 6 - i$$

9.
$$\begin{aligned}\left(\frac{-6+2i}{1-8i}\right)^2 &= \left(\frac{(-6+2i)(1+8i)}{(1-8i)(1+8i)}\right)^2 \\ &= \frac{(-22-46i)^2}{65^2} = \frac{1}{4225}(-1632+2024i)\end{aligned}$$

In each of Problems 11–16, n denotes an arbitrary integer.

11.
$$|3i| = 3, \arg(3i) = \frac{\pi}{2} + 2n\pi$$

13.
$$|-3+2i| = \sqrt{13}, \arg(-3+2i) = -\arctan(2/3) + (2n+1)\pi$$

15.
$$|-4| = 4, \arg(-4) = (2n+1)\pi$$

17. Because $|-2+2i| = 2\sqrt{2}$ and $3\pi/34$ is an argument, the polar form of $-2+2i$ is
$$-2+2i = 2\sqrt{2}e^{3i\pi/4}.$$
Here we did not add the customary $2n\pi$ to the argument because, first, we need only one argument to write the polar form, and second, $e^{2n\pi i} = 1$ for any integer n.

19. $|5-2i| = \sqrt{29}$ and an argument of $5-2i$ is $-\arctan(2/5)$, so
$$5-2i = \sqrt{29}e^{-\arctan(2/5)i}.$$

21.
$$8+i = \sqrt{65}e^{\arctan(1/8)i}.$$

23. Because $i^2 = -1$, we have
$$\begin{aligned} i^{4n} &= (i^2)^{2n} = ((-1)^2)^n = 1, \\ i^{4n+1} &= i^{4n}i = i, \\ i^{4n+2} &= i^{4n}i^2 = i^2 = -1, \\ i^{4n+3} &= i^{4n}i^3 = i^2 i = -i. \end{aligned}$$

25. Suppose first that z, w, u form vertices of a triangle, labeled in clockwise order around the triangle. The sides of the triangle are vectors represented by the complex numbers $w-z$, $u-w$, and $z-u$. This triangle is equilateral if and only if the sides have the same length, or
$$|w-z| = |u-w| = |z-u|$$
and each of the vector sides can be rotated by $\theta = 2\pi/3$ radians clockwise to coincide with another side. This occurs exactly when
$$u-w = (w-z)e^{-2\pi i/3} \text{ and } z-u = (u-w)e^{-2\pi i/3}.$$
Dividing these equations, we have
$$\frac{u-w}{z-u} = \frac{w-z}{u-w}.$$
Then
$$(u-w)(u-w) = (w-z)(z-u).$$
Then
$$u^2 - 2uw + w^2 = wz + zu - uw - z^2.$$
Then
$$z^2 + w^2 + u^2 = zw + zu + wu.$$

27. Suppose first that $|z| = 1$. Then

$$|\overline{z}| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z| = 1$$

also. Then

$$\begin{aligned}\left|\frac{z-w}{1-\overline{z}w}\right| &= \left|\frac{z-w}{z\overline{z}-\overline{z}w}\right| \\ &= \frac{|z-w|}{|\overline{z}||z-w|} = 1.\end{aligned}$$

If $|w| = 1$, then

$$\begin{aligned}\left|\frac{z-w}{1-\overline{z}w}\right| &= \left|\frac{z-w}{\overline{w}w-\overline{z}w}\right| \\ &= \frac{1}{|\overline{w}|}\left|\frac{z-w}{\overline{z}-\overline{w}}\right| = 1\end{aligned}$$

because

$$|z-w| = |\overline{w-z}| = |\overline{w} - \overline{z}|.$$

29. M consists of all $x + iy$ with $y < 7$. This is the half-plane lying below the horizontal line $y = 7$. The boundary points are all points $x + 7i$ on the "edge" of M. M is open because it does no contain any of its boundary points (all points of M are interior points).

31. U consists of all points in the vertical strip between the vertical lines $x = 1$ and $x = 3$, including points on the line $x = 3$, but none of the points on the line $x = 1$. The boundary points of U are the points $1 + iy$ and $3 + iy$ on these lines. U is not closed because there are boundary points of U that do not belong to U. U is not open because U contains some of its boundary points (so not every point of U is an interior point).

33. W consists of all $x+iy$ with $x > y^2$. These are the points "enclosed" by the parabola $x = y^2$, which opens to the right from the origin. The boundary points are the points on the parabola, which are the points $x + i\sqrt{x}$ for $x \geq 0$. W does not contain any of its boundary points, and is open. W is not closed.

19.2 Complex Functions

1.
$$f(z) = z - i = x + iy - i = x + (y-1)i$$

so $u(x, y) = x$ and $v(x, y) = y - 1$. The Cauchy-Riemann equations for this function are

$$\frac{\partial u}{\partial x} = 1 = \frac{\partial v}{\partial y}$$

and
$$\frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x}.$$
Because u and v are continuous with continuous first partial derivatives, and the Cauchy-Riemann equations are satisfied for all z, $f(z)$ is differentiable for all z.

3. $f(z) = |x + iy| = \sqrt{x^2 + y^2}$, so
$$u(x, y) = \sqrt{x^2 + y^2} \text{ and } v(x, y) = 0.$$
If $z \neq 0$, then
$$\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$
and
$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0.$$
The Cauchy-Riemann equations are not satisfied at any nonzero z. To check what happens at $z = 0$, compute
$$\begin{aligned}\frac{\partial u}{\partial x}(0, 0) &= \lim_{h \to 0} \frac{u(h, 0) - u(0, 0)}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{h^2}}{h} \\ &= \lim_{h \to 0} \frac{|h|}{h}.\end{aligned}$$
This limit does not exist, because
$$\frac{|h|}{h} = \begin{cases} 1 & \text{if } h > 0, \\ -1 & \text{if } h < 0. \end{cases}$$
Similarly, $(\partial u/\partial y)(0, 0)$ does not exist. Therefore the Cauchy-Riemann equations are not satisfied at any point, including the origin, and $f(z)$ is not differential for any z.

5. $f(z) = i|z|^2 = (x^2 + y^2)i$, so
$$u(x, y) = 0 \text{ and } v(x, y) = x^2 + y^2.$$
Then
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$
and
$$\frac{\partial v}{\partial x} = 2x, \frac{\partial v}{\partial y} = 2y.$$

The Cauchy-Riemann equations are satisfied only at $z = 0$, so $f(z)$ is certainly not differentiable at any nonzero z. To check at $z = 0$, fall back on the definition of the derivative:

$$\lim_{h\to 0} \frac{z(h) - f(0)}{h} = \lim_{h\to 0} \frac{i|h|^2}{h} = \lim_{h\to 0} \frac{ih\overline{h}}{h} = \lim_{h\to 0} i\overline{h} = 0.$$

Therefore $f'(0) = 0$. 0 is the only point at which this function is differentiable.

7. First,

$$f(z) = \frac{x + iy}{x} = 1 + \frac{y}{x}i$$

for $x \neq 0$. This function is defined for all z except for points on the imaginary axis. For $x \neq 0$, we can let

$$u(x, y) = 1, v(x, y) = \frac{y}{x}.$$

Now

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$

and

$$\frac{\partial v}{\partial x} = -\frac{y}{x^2}, \frac{\partial v}{\partial y} = \frac{1}{x}.$$

These are not satisfied at any z at which the function is defined. Therefore $f(z)$ is not differentiable at any point at which it is defined.

9. First,

$$f(z) = (\overline{z})^2 = (x - iy)^2 = x^2 - y^2 - 2xyi,$$

so let

$$u(x, y) = x^2 - y^2 \text{ and } v(x, y) = -2xy.$$

Then

$$\frac{\partial u}{\partial x} = 2x \text{ but } \frac{\partial v}{\partial y} = -2x,$$

while

$$\frac{\partial v}{\partial x} = -2y = \frac{\partial u}{\partial y}.$$

The Cauchy-Riemann equations hold only a $z = 0$, so this is the only point at which $f(z)$ might have a derivative. To check this, look at

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{(\overline{h})^2}{h} = \lim_{h\to 0} \left(\frac{\overline{h}}{h}\right) \overline{h} = 0$$

because $\overline{h}/h$ has magnitude 1 and $\overline{h} \to 0$ if $h \to 0$.

Therefore $f'(0) = 0$, and 0 is the only point at which the function has a derivative.

11. For $z \neq 0$, write

$$f(z) = -4z + \frac{1}{z} = -4x - 4iy + \frac{1}{x+iy}$$
$$= -4x - 4yi + \frac{x-iy}{x^2+y^2}$$

for $(x, y) \neq (0, 0)$. Let

$$u(x, y) = -4x + \frac{x}{x^2+y^2} \text{ and } v(x, y) = \left(-4y - \frac{y}{x^2+y^2}\right).$$

Then

$$\frac{\partial u}{\partial x} = -4 + \frac{y^2-x^2}{(x^2+y^2)^2}, \frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2},$$

and

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}, \frac{\partial v}{\partial y} = -4 + \frac{y^2-x^2}{(x^2+y^2)^2}.$$

The Cauchy-Riemann equations are satisfied at each nonzero z. Because u, v and the partial derivatives are continuous for $(x, y) \neq (0, 0)$, $f(z)$ is differentiable for all nonzero z.

13. Let $z_n = x_n + iy_n$ and $z_0 = x_0 + iy_0$. Write $f(z) = u(x, y) + iv(x, y)$. Because u and v are continuous at (x_0, y_0), then

$$f(z_n) = u(x_n, y_n) + iv(x_n, y_n) \to u(x_0, y_0) + iv(x_0, y_0) = f(z_0).$$

19.3 The Exponential and Trigonometric Functions

1.

$$e^i = e^{0+i} = e^0(cos(1) + i\sin(1)) = \cos(1) + i\sin(1)$$

3. Use the fact that

$$\cos(x + iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

to get

$$\cos(3 + 2i) = cos(3)\cosh(2) - i\sin(3)\sinh(2).$$

5.

$$e^{5+2i} = e^3e^{2i} = e^3\cos(2) + ie^3\sin(2).$$

7.

$$\sin^2(1 + i) = \frac{1}{2}(1 - \cos(2(1 + i)))$$
$$= \frac{1}{2}[1 - \cos(2)\cosh(2) + i\sin(2)\sinh(2)].$$

9.
$$e^{i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = i$$

11. Begin with
$$\begin{aligned} e^{z^2} &= e^{x^2-y^2+2ixy} \\ &= e^{x^2-y^2}[\cos(2xy) + i\sin(2xy)]. \end{aligned}$$

Then
$$u(x,y) = e^{x^2-y^2}\cos(2xy) \text{ and } v(x,y) = e^{x^2-y^2}\sin(2xy).$$

Now compute
$$\begin{aligned} \frac{\partial u}{\partial x} &= e^{x^2-y^2}[2x\cos(2xy) - 2y\sin(2xy)], \\ \frac{\partial u}{\partial y} &= e^{x^2-y^2}[-2y\cos(2xy) - 2x\sin(2xy)], \\ \frac{\partial v}{\partial x} &= e^{x^2-y^2}[2x\sin(2xy) + 2y\sin(2xy)], \\ \frac{\partial v}{\partial y} &= e^{x^2-y^2}[-2y\sin(2x) + 2x\cos(2xy)]. \end{aligned}$$

Then u and v satisfy the Cauchy-Riemann equations for all (x,y).

13.
$$\begin{aligned} f(z) = ze^z &= (x+iy)e^x(\cos(y) + i\sin(y)) \\ &= xe^x\cos(y) - ye^x\sin(y) + (ye^x\cos(y) + xe^x\sin(y))i = u(x,y) + iv(x,y). \end{aligned}$$

Then
$$\frac{\partial u}{\partial x} = e^x[\cos(y) + x\cos(y) - y\sin(y)] = \frac{\partial v}{\partial y}$$
and
$$\frac{\partial u}{\partial y} = e^x[-x\sin(y) - \sin(y) - y\cos(y)] = -\frac{\partial v}{\partial x}.$$
The Cauchy-Riemann equations are satisfied for all z.

15. Suppose $e^z = 2i$. With $z = x + iy$, then
$$e^x\cos(y) + ie^x\sin(y) = 2i.$$

Then
$$e^x\cos(y) = 0 \text{ and } e^x\sin(y) = 2.$$
Because $e^x \neq 0$, $\cos(y) = 0$, so
$$y = \frac{(2n+1)\pi}{2}$$

in which n can be any integer. Now we have

$$e^x \sin\left(\frac{2n+1}{2}\pi\right) = 2.$$

Now $e^x > 0$ for real x, so $\sin((2n+1)\pi/2) > 0$. But

$$\sin\left(\frac{2n+1}{2}\pi\right) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ -1 & \text{if } n \text{ is odd.} \end{cases}$$

Therefore n must be even, say $n = 2m$. Now we have

$$y = \frac{4m+1}{2}\pi.$$

Now we are left with $e^x = 2$, so $x = \ln(2)$. All the solutions of $e^z = 2i$ are

$$\ln(2) + \frac{4m+1}{2}\pi$$

with m any integer.

17. Use the polar form of the given equation. If $z = re^{i\theta}$, the equation is

$$e^z = e^r e^{i\theta} = -2.$$

Because θ is real, $|e^{i\theta}| = 1$, so

$$|e^z| = e^r = |-2| = 2.$$

Then $r = \ln(2)$. Next, we must also have

$$e^{i\theta} = -1 = \cos(\theta) + i\sin(\theta).$$

Then $\sin(\theta) = 0$, so $\theta = n\pi$, in which (so far) n can be any integer. But $\cos(\theta) = -1$ means that n must be odd, so

$$\theta = (2m+1)\pi$$

in which m can be an y integer. Then

$$z = \ln(2) + (2m+1)\pi,$$

with m any integer, are all the solutions for z.

19.4 The Complex Logarithm

1. In polar form,

$$z = -4i = 4e^{3n\pi i/2}$$

so

$$\log(-4i) = \ln(4) + \left(\frac{\pi}{2} + 2n\pi\right)i.$$

3. $-5 = 5e^{\pi i}$, so
$$\log(-5) = \ln(5) + (2n+1)\pi i.$$

5. $-9 + 2i = \sqrt{85}e^{(\arctan(-2/9)+\pi)i}$, so
$$\log(-9+2i) = \frac{1}{2}\ln(85) + (-\arctan(2/9) + (2n+1)\pi)i.$$

7. Note that $\log(zw), \log(z)$ and $\log(w)$ all have infinitely many different values, so we cannot expect to write the complex logarithm of the product as the sum of the logarithms of the factors. What we can show is that every value of $\log(zw)$ is the sum of a value of $\log(z)$ and a value of $\log(w)$. Suppose that z and w are nonzero. Let θ_z be any argument of z, and θ_w any argument of w. Then
$$z = |z|e^{(\theta_z+2n\pi)i} \text{ and } w = |w|e^{(\theta_w+2m\pi)i}.$$
Then
$$zw = |z||w|e^{(\theta_z+\theta_w+2k\pi)i},$$
while
$$\log(z) + \log(w) = \ln(|z|) + \ln(|w|) + (\theta_z + \theta_w + 2(n+m)\pi)i.$$
This means that for any choice of n and m, we can choose $k = n + m$ to obtain a value of $\log(zw)$ that is equal to $\log(z) + \log(w)$.

19.5 Powers

In these problems, n denotes an arbitrary integer.

1.
$$\begin{aligned} i^{1+i} &= e^{(1+i)\log(i)} = e^{(1+i)(\pi/2+2n\pi)i} \\ &= e^{-(\pi/2+2n\pi)}\left[\cos\left(\frac{\pi}{2}+2n\pi\right) + i\sin\left(\frac{\pi}{2}+2n\pi\right)\right] \\ &= ie^{-(\pi/2+2n\pi)}. \end{aligned}$$

3.
$$\begin{aligned} i^i &= e^{i\log(i)} = e^{i(i(\pi/2+2n\pi))} \\ &= e^{-\pi/2+2n\pi}. \end{aligned}$$
This is consistent with Problem 1, because $i^{1+i} = ii^i$.

5.
$$\begin{aligned} (-1+i)^{-3i} &= e^{-3i\log(-1+i)} \\ &= e^{-3i\ln(\sqrt{2})+i(3\pi/4+2n\pi)} \\ &= e^{9\pi/4+6n\pi}[\cos(3\ln(\sqrt{2})) + i\sin(3\ln(\sqrt{2}))]. \end{aligned}$$

7.

$$\begin{aligned} i^{1/4} &= \left(e^{i(\pi/2+2n\pi)}\right)^{1/4} \\ &= e^{i(\pi/8+n\pi/2)}, \end{aligned}$$

the the four fourth roots obtained for $n = 0, 1, 2, 3$. Other choices of n repeat these roots.

9.

$$\begin{aligned} (-4)^{2-i} &= e^{(2-i)\log(-4)} = e^{(2-i)(\ln(4)+i(\pi+2n\pi))} \\ &= e^{2\ln(4)+\pi+2n\pi}[\cos(\ln(4)) - i\sin(\ln(4))]. \end{aligned}$$

11.

$$\begin{aligned} (-16)^{1/4} &= \left(16e^{i(\pi+2n\pi)}\right)^{1/4} = 2e^{i(\pi/4+n\pi/2)} \\ &= 2\left[\cos\left(\frac{\pi}{4} + \frac{n\pi}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)\right]. \end{aligned}$$

We obtain the four fourth roots by taking $n = 0, 1, 2, 3$. These fourth roots are

$$\sqrt{2}(1+i), \sqrt{2}(-1+i), \sqrt{2}(-1-i), \sqrt{2}(1-i).$$

13. These are the sixth roots of unity:

$$\begin{aligned} 1^{1/6} &= \left(e^{2n\pi i}\right)^{1/6} = e^{n\pi i/3} \\ &= \cos(n\pi/3) + i\sin(n\pi/3). \end{aligned}$$

These sixth roots are obtained for $n = 0, 1, 2, 3, 4, 5$, and are

$$1, \frac{1}{2}(1+\sqrt{3}i), \frac{1}{2}(-1+\sqrt{3}i), -1, \frac{1}{2}(-1-\sqrt{3}i), \frac{1}{2}(1-\sqrt{3}i).$$

15. Let ω be any nth root of 1 different from 1. The numbers ω^j, for $j = 0, 1, \cdots, n-1$ are distinct, hence are all of the nth roots of 1. It is therefore enough to show that

$$\sum_{j=0}^{n-1} \omega^j = 0.$$

But this is a finite geometric series, whose sum is known:

$$\sum_{j=0}^{n-1} \omega^j = \frac{1-\omega^n}{1-\omega} = 0$$

because $\omega^n = 1$.

The conclusion can also be proved as follows. Let $\omega_1, \cdots, \omega_n$ be the nth roots of unity. Let $S = \sum_{j=1}^{n} \omega_j$.

Now, one of the nth roots of unity is 1, but the other $n-1$ roots are different from 1. Pick one root that does not equal 1, say, possibly by relabeling, $\omega_1 \neq 1$. The numbers

$$\omega_1\omega_1, \omega_1\omega_2, \cdots, \omega_1\omega_n$$

are also nth roots of unity and are distinct, so this list includes all the nth roots of unity. The sum of these numbers is therefore S:

$$S = \sum_{j=1}^{n} \omega_1\omega_j = \omega_1 S.$$

But then

$$S(1-\omega_1) = 0.$$

Because $\omega_1 \neq 1$, then $S = 0$.

Chapter 20

Complex Integration

20.1 The Integral of a Complex Function

1. In this problem $f(z) = 1$ is differentiable for all z and we can write an antiderivative $F(z) = z$. The curve has initial point $\gamma(1) = 1 - i$ and terminal point $\gamma(3) = 9 - 3i$, so

$$\int_\gamma f(x)\,dz = F(9-3i) - F(1-i) = 9 - 3i - (1-i) = 8 - 2i.$$

We can also evaluate the integral by using the parametric equations of the curve. On γ, $z = \gamma(t) = t^2 - it$, so $dz = 2t - i$ and

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_\gamma dz \\ &= \int_1^3 (2t-i)\,dt \\ &= t^2 - it\Big|_1^3 = (9-3i) - (1-i) \\ &= 8 - 2i.\end{aligned}$$

3. $f(z) = \operatorname{Re}(z)$ is not differentiable, so there is no antiderivative. There are many ways to parametrize the curve. One is by setting

$$\gamma(t) = 1 + (1+i)t \text{ for } 0 \leq t \leq 1.$$

On γ, $f(z) = 1 + t$ and $dz = (1 + i)\,dt$, so

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_0^1 (1+t)(1+i)\,dt \\
&= \int_0^1 (1+i+(1+i)t)\,dt \\
&= (1+i)t + \frac{1+i}{2}t^2\Big|_0^1 \\
&= \frac{3}{2}(1+i).
\end{aligned}$$

5. $F(z) = (z-1)^2/2$ is an antiderivative of $f(z)$, which is differentiable for all z, so

$$\int_\gamma f(z)\,dz = F(1-4i) - F(2i) = -\frac{13}{2} + 2i.$$

7. $f(z)$ is differentiable for all z and has antiderivative $F(z) = -\cos(2z)/2$, so

$$\begin{aligned}
\int_\gamma f(z)\,dz &= F(-4i) - F(-i) \\
&= -\frac{1}{2}(\cos(-8i) - \cos(-2i)) = -\frac{1}{2}[\cosh(8) - \cosh(2)].
\end{aligned}$$

9. $f(z)$ is differentiable for all z and has antiderivative $F(z) = -i\sin(z)$, so

$$\begin{aligned}
\int_\gamma f(z)\,dz &= F(2+i) - F(0) = -i\sin(2+i) \\
&= -i[\sin(2)\cosh(1) + i\cos(2)\sinh(1)] \\
&= -\cos(2)\sinh(1) - i\sin(2)\cosh(1).
\end{aligned}$$

11. Use the antiderivative $F(z) = (z-i)^4/4$ to get

$$\int_\gamma f(z)\,dz = F(2-4i) - F(0) = 10 + 210i.$$

13. $f(z)$ has no antiderivative because this function is not differentiable. Parametrize the curve by $\gamma(t) = (-4+3i)t$ for $0 \le t \le 1$. Then

$$\begin{aligned}
\int_\gamma i\overline{z}\,dz &= \int_0^1 -i(4t - 3ti)(-4+3i)\,dt \\
&= (-4+3i)\left(\frac{3}{2} - 2i\right) = \frac{25}{2}i.
\end{aligned}$$

15. $f(z) = |z|^2$ has no antiderivative, so write $\gamma(t) = (1+i)t - i$ for $0 \le t \le 1$ to get

$$\int_\gamma |z|^2\,dz = \int_0^1 (t^2 + (t-1)^2)(1+i)\,dt = \frac{2}{3}(1+i).$$

17. The length of γ is $\sqrt{5}$. Now we need a number M such that

$$\left|\frac{1}{1+z}\right| \le M \text{ on } \gamma.$$

Notice that the point on γ closest to $z = -1$ is $2+i$, so for z on the curve,

$$|z+i| = |z-(-1)| \ge |2+i+i| = \sqrt{10}.$$

Then

$$\left|\frac{1}{1+z}\right| = \frac{1}{|1+z|} \le \frac{1}{\sqrt{10}}.$$

We can therefore use $M = 1/\sqrt{10}$ to get the bound

$$\left|\int_\gamma \frac{1}{1+z}\,dz\right| \le \frac{\sqrt{5}}{\sqrt{10}} = \frac{1}{\sqrt{2}}.$$

20.2 Cauchy's Theorem

1. $\sin(z)$ is differentiable for all z, hence on an open set containing the curve and all points enclosed by the curve. By Cauchy's theorem,

$$\oint_\gamma \sin(z)\,dz = 0.$$

3. γ encloses $2i$, at which $f(z)$ is not defined. Parametrize

$$\gamma(t) = 2i + 2e^{it} \text{ for } 0 \le t \le 2\pi.$$

Then

$$\begin{aligned}\oint_\gamma \frac{1}{(z-2i)^3}\,dt &= \int_0^{2\pi} \frac{1}{(2e^{it})^3} 2ie^{it}\,dt \\ &= \frac{i}{4}\int_0^{2\pi} e^{-2it}\,dt = 0.\end{aligned}$$

This integral happens to be zero, but we could not conclude this from Cauchy's theorem, which does not apply here.

5. $f(z) = \overline{z}$ is not differentiable. Write $\gamma(t) = e^{it}$ for $0 \le t \le 2\pi$. Then

$$\oint_\gamma \overline{z}\,dz = \int_0^{2\pi} e^{-it} ie^{it}\,dt = 2\pi i.$$

7. Because $f(z) = ze^z$ is differentiable on the curve and throughout the region it encloses, then by Cauchy's theorem,

$$\oint_\gamma ze^z\,dz = 0.$$

9. $f(z) = |z|^2$ is not differentiable at any point other than 0, Cauchy's theorem does not apply. Write $\gamma(t) = 7e^{it}$ for $0 \le t \le 2\pi$ to obtain

$$\oint_\gamma |z|^2\,dz = \int_0^{2\pi} 49(7ie^{it})\,dt = 0.$$

11. $f(z) = \mathrm{Re}(z)$ is not differentiable, so write $\gamma(t) = 2e^{it}$ for $0 \le t \le 2\pi$. Then

$$\begin{aligned}\oint_\gamma \mathrm{Re}(z)\,dz &= \int_0^{2\pi} 2\cos(t)(2ie^{it})\,dt \\ &= \int_0^{2\pi} [4i\cos^2(t) - 4\cos(t)\sin(t)]\,dt = 4\pi i.\end{aligned}$$

20.3 Consequences of Cauchy's Theorem

1. Because $2i$ is the center of the circle γ, we can apply Cauchy's integral formula with $f(z) = z^4$ to obtain

$$\oint_\gamma \frac{z^4}{z-2i}\,dz = 2\pi i f(2i) = 2\pi i(2i)^4 = 32\pi i.$$

3. Use Cauchy's integral formula, with $f(z) = z^2 - 4z + i$, to obtain

$$\begin{aligned}\oint_\gamma \frac{z^2-4z+i}{z-1+2i}\,dz &= 2\pi i f(1-2i) \\ &= 2\pi i[(1-2i)^2 - 4(1-2i) + i] = 2\pi i(-8+7i) \\ &= -14\pi - 16\pi i.\end{aligned}$$

5. We can use the Cauchy integral formula for derivatives with $n = 1$ and $f(z) = ie^z$:

$$\begin{aligned}\oint_\gamma \frac{ie^z}{(z-2+i)^2}\,dz &= 2\pi i f'(2-i) \\ &= 2\pi i(ie^{2-i}) = -2\pi e^2[\cos(1) - i\sin(1)].\end{aligned}$$

7. With $f(z) = z\sin(3z)$ and $n = 2$, Cauchy's formula for derivatives gives us

$$\begin{aligned}\oint_\gamma \frac{z\sin(3z)}{(z+4)^3}\,dz &= \frac{2\pi i}{2} f''(-4) \\ &= \pi i[6\cos(12) - 36\sin(12)].\end{aligned}$$

9.

$$\begin{aligned}
\oint_\gamma \frac{-(2+i)\sin(z^4)}{(z+4)^2}\,dz &= -2\pi i(2+i)\frac{d}{dz}(\sin(z^4))\Big|_{z=-4} \\
&= 2\pi i(1-2i)\left[4z^3\cos(z^4)\right]_{z=-4} \\
&= -512\pi(1-2i)\cos(256).
\end{aligned}$$

11. Parametrize $\gamma(t) = 3 - t + (1-6t)i$ for $0 \le t \le 1$. Then

$$\begin{aligned}
\int_\gamma \mathrm{Re}(z+4)\,dz &= \int_0^1 (7-t)(-1-6i)\,dt \\
&= (-1-6i)\frac{13}{2} = -\frac{13}{2} - 39i.
\end{aligned}$$

13. First evaluate

$$\oint_\gamma \frac{e^z}{z}\,dz$$

by Cauchy's integral formula to obtain

$$\int_\gamma \frac{e^z}{z}\,dz = 2\pi i e^z\Big|_{z=0} = 2\pi i.$$

Now evaluate this integral by parametrizing $\gamma(t) = e^{it}$ for $0 \le t \le 2\pi$:

$$\begin{aligned}
\oint_\gamma \frac{e^z}{z}\,dz &= \int_0^{2\pi} \frac{e^{\cos(t)+i\sin(t)}}{e^{it}} ie^{it}\,dt \\
&= i\int_0^{2\pi} e^{\cos(t)}\cos(\sin(t))\,dt - \int_0^{2\pi} e^{\cos(t)}\sin(\sin(t))\,dt \\
&= 2\pi i.
\end{aligned}$$

By equating the real parts of both sides of this equation, and then the imaginary parts, we obtain

$$\int_0^{2\pi} e^{\cos(t)}\cos(\sin(t))\,dt = 2\pi$$

and

$$\int_0^{2\pi} e^{\cos(t)}\sin(\sin(t))\,dt = 0.$$

The first integral is not obvious. The second could be done without complex analysis by observing that the integral from 0 to π is the negative of the integral from π to 2π.

Chapter 21

Series Representations of Functions

21.1 Power Series

In each of Problems 1–6, the strategy is to take the limit of the magnitude of the ratio of successive terms of the series. The series converges when this limit (if it exists) is less than 1.

1. Take the limit of the magnitude of successive terms:

$$\left|\frac{(n+2)/2^{n+1}}{(n+1)/2^n}|z+3i| = \frac{1}{2}\frac{n+2}{n+1}(z+3i)\right.$$
$$\to \frac{1}{2}|z+3i|.$$

The series converges (absolutely) if

$$\frac{1}{2}|z+3i| < 1$$

or

$$|z+3i| < 2.$$

The power series has radius of convergence 2 and open disk of convergence $|z+3i| < 2$, the open disk of radius 2 about the center $-3i$.

3.

$$\begin{aligned}
&\left|\frac{(n+1)^{n+1}/(n+2)^{n+1}}{n^n/(n+1)^n}(z-1+3i)\right| \\
&= \frac{(n+1)^{2n+1}}{n^n(n+2)^{n+1}}|z-1+3i| \\
&= \left(\frac{n+1}{n}\right)^n\left(\frac{n+1}{n+2}\right)^{n+1}|z-1+3i| \\
&= \left(1+\frac{1}{n}\right)^n\left(\frac{1+1/n}{1+2/n}\right)^n\left(\frac{1+1/n}{1+2/n}\right)|z-1+3i|
\end{aligned}$$

and the limit of this quantity is less than 1 if $|z-1+3i|<1$. The radius of convergence is 1 and the disk of convergence is the open disk of radius 1 centered at $1-3i$.

In this limit, we have used (several times) the fact that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x.$$

5.

$$\left|\frac{i^{n+1}/2^{n+2}}{i^n/2^n}(z+8i)\right| \to \frac{1}{2}|z+8i|$$

This ratio has limit <1 if $|z+8i|<2$. The power series has radius of convergence 2 and the open disk of convergence is the open disk of radius 2 centered at $-8i$.

7. No. The power series has center $2i$. If the series converges at 0, it must also converge at the point i that is closer to the center $2i$ than 0 is.

In each of Problems 9–14, we attempt to use known series to derive the requested series.

9. Assuming that we know the series for $\cos(z)$:

$$\cos(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}.$$

This converges for all z. Replace z with $2z$ to obtain

$$\cos(2z) = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}}{(2n)!}z^{2n}.$$

In writing these series, be careful with factorials. For example, in general $(2n)! \neq 2n!$.

11. This is just a rearrangement of the given polynomial into powers of $z-2+i$. This can be done algebraically, or we can write the Maclaurin series of this polynomial about $2-i$. This series will be

$$f(z) = z^2 - 3z + i = c_0 + c_1(z-2+i) + z_2(z-2+i)^2,$$

where

$$c_0 = f(2-i) = -3, c_1 = f'(2-i) = 1-2i$$

and

$$c_2 = \frac{1}{2}f''(2-i) = 1.$$

The expansion of $f(z)$ about $2-i$ is

$$z^2 - 3z + i = -3 + (1-2i)(z-2+i) + (z-2+i)^2.$$

13. Like Problem 11, this can be done as an algebraic rearrangement of terms in $f(z) = (z-9)^2$, or as a power series about $1+i$, which will be in powers of $z-i-i$. Using the latter approach, compute the coefficients

$$c_0 = f(1+i) = 63 - 16i, c_1 = f'(1+i) = -16 + 2i$$

and

$$c_2 = \frac{1}{2}f''(1+i) = 1.$$

Then

$$(z-9)^2 = 63 - 16i + (-16+2i)(z-1-i) + (z-1-i)^2.$$

15. We know that $f(0) = 1$, $f'(0) = i$, and $f''(z) = 2f(z) + 1$. Compute

$$\begin{aligned} f''(0) &= 2f(0) + 1 = 3, \\ f^{(3)}(0) &= 2f''(0) = 2i, \\ f^{(4)}(0) &= 2f''(0) = 6, \\ f^{(5)}(0) &= 2f^{(3)}(0) = 4i. \end{aligned}$$

Now use Taylor's formula for the coefficients (in this case, about 0,

$$c_n = \frac{1}{n!}f^{(n)}(0)$$

to write the first six terms of the expansion:

$$1 + iz + \frac{3}{2}z^2 + \frac{2i}{3!}z^3 + \frac{6}{4!}z^4 + \frac{4i}{5!}z^5.$$

In this problem it is not difficult to write the entire Maclaurin expansion, because an inductive argument shows that

$$f^{(2n)}(0) = 2^n + 2^{n-1} \text{ and } f^{(2n+1)}(0) = 2^n i.$$

17. Let z be a complex number and consider the integral

$$\frac{1}{2\pi i}\oint_\gamma \frac{z^n}{n!w^{n+1}}e^{zw}\,dw.$$

Here γ is the unit circle about the origin, oriented counterclockwise as usual. Expand e^{zw} in its Maclaurin series and parametrize $\gamma(t) = e^{it}$ for $0 \leq t \leq \pi$ to write

$$\begin{aligned}
\oint_\gamma \frac{z^n}{n!w^{n+1}}e^{zw}\,dw &= \frac{1}{2\pi i}\oint_\gamma \frac{z^n}{n!w^{n+1}}\sum_{k=0}^{\infty}\frac{(zw)^k}{k!}\,dw \\
&= \frac{1}{2\pi i}\oint_\gamma \sum_{k=0}^{\infty}\frac{z^{n+k}w^{k-n-1}}{n!k!}\,dw \\
&= \frac{1}{2\pi i}\int_0^{2\pi}\sum_{k=0}^{\infty}\frac{z^{n+k}}{e}^{i(k-n-1)t}\qquad n!k!ie^{it}\,dt \\
&= \sum_{k=0}^{\infty}\frac{1}{2\pi}\int_0^{2\pi}\frac{z^{n+k}}{n!k!}e^{i(k-n)t}\,dt.
\end{aligned}$$

Now,

$$\int_0^{2\pi} e^{i(k-n)t}\,dt = \begin{cases} 0 & \text{if } k \neq n, \\ 2\pi & \text{if } k = n. \end{cases}$$

We therefore have

$$\frac{1}{2\pi i}\oint_\gamma \frac{z^n}{n!w^{n+1}}e^{zw}\,dw = \frac{(z^n)^2}{(n!)^2}.$$

Finally, we can write

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{1}{(n!)^2}z^{2n} &= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\frac{z^n}{n!w^{n+1}}e^{zw}\,dw \\
&= \frac{1}{2\pi i}\int_0^{2\pi}\sum_{n=0}^{\infty}\frac{z^n}{n!e^{i(n+1)t}}e^{ze^{it}}e^{it}\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{n=0}^{\infty}\frac{(ze^{-it})^n}{n!}\right]e^{ze^{it}}\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{ze^{-it}}e^{ze^{it}}\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{z(e^{it}+e^{-it})}\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{2z\cos(t)}\,dt.
\end{aligned}$$

19. $f(z)$ has a zero of order 4 at 0 because z^2 has a zero of order 2 at 0 and $\sin^2(z)$ also has a zero of 2 there (because $\sin(z)$ has a zero of order 1 at 0).

21. $f(z)$ has a zero of order 3 at $3\pi/2$ because $\cos(z)$ has a simple zero there.

23. $f(z)$ is not defined at $z = 0$, so we cannot really speak of it having a zero there. However, notice something interesting. Using the Maclaurin expansion of $\sin(z)$, with z^4 in place of z, and divided by z^2, we can write

$$\frac{1}{z^2}\sin(z^4) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{8n+2}.$$

This power series converges for all z, and has the value 0 at 0. We can therefore extend $f(z)$ by giving it the value 0 at $z = 0$, and obtain a differentiable function. This extended function has a zero of order 2 at 0.

25. Compute the kth derivative of $f(z)$ at z_0 using each series, obtaining

$$\begin{aligned} f^k(z_0) &= \sum_{n=0}^{\infty} a_n(n)(n-1)\cdots(n-k+1)(z-z_0)^{n-k} \\ &= \sum_{n=0}^{\infty} b_n(n)(n-1)\cdots(n-k+1)(z-z_0)^{n-k}. \end{aligned}$$

Then

$$f^k(z_0) = k!a_k = k!b_k,$$

so for each $k = 0, 1, 2, \cdots$,

$$a_k = \frac{1}{k!} f^{(k)}(z_0) = b_k.$$

21.2 The Laurent Expansion

Problems 1–10 are solved using manipulations of known series, such as geometric series and power series for exponential and trigonometric functions. It is sometimes best, in seeking an expansion about z_0, to focus on getting an expression involving powers of $z - z_0$, using algebraic manipulations, or sometimes integration and differentiation.

In particular, it is useful to know the geometric series

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$$

and

$$\frac{1}{1+r} = \sum_{n=0}^{\infty} (-1)^n r^n,$$

for $|r| < 1$.

1. We want an expansion in powers of $z-i$. To this end, begin with

$$\frac{2z}{1+z^2} = \frac{1}{z-i} + \frac{1}{z+i}.$$

The first term is already an expansion in powers of $z-i$ (having only one term). For the second term, write

$$\begin{aligned}\frac{1}{z+i} &= \frac{1}{2i+(z-i)} = \frac{1}{2i\left(1-\frac{z-i}{2i}\right)} \\ &= \frac{1}{2i}\sum_{n=0}^{\infty}(-)^n\left(\frac{z-i}{2i}\right)^n \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2i)^{n+1}}(z-i)^n.\end{aligned}$$

This expansion is valid for

$$\left|\frac{z-i}{2i}\right| = \frac{1}{2}|z-i| < 1,$$

or

$$|z-i| < 2.$$

The Laurent expansion of $f(z)$ about i is therefore

$$\frac{1}{z-i} + \sum_{n=0}^{\infty}\frac{(-1)^n}{(2i)^{n+1}}(z-i)^n.$$

This represents $f(z)$ in the annulus $0 < |z-i| < 2$.

3. If $z \neq 0$, then

$$\begin{aligned}\frac{1-\cos(2z)}{z^2} &= \frac{1}{z^2}\left[1-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(2z)^{2n}\right] \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}4^n}{(2n)!}z^{2n-2}.\end{aligned}$$

5. The denominator is already in terms of $z-1$, so concentrate on the numerator:

$$\begin{aligned}\frac{z^2}{1-z} &= \frac{((z-1)+1)^2}{1-z} = -\frac{1+2(z-1)+(z-1)^2}{z-1} \\ &= -\frac{1}{z-1} - 2 - (z-1).\end{aligned}$$

This represents the function for $0 < |z-1| < \infty$, the complex plane with 1 removed.

7. Use the exponential series to obtain

$$\frac{1}{z^2}e^{z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{1}{n!}z^{2n} = \sum_{n=0}^{\infty}\frac{1}{n!}z^{2n-2},$$

for $0 < |z| < \infty$.

9. The denominator is already a power of $z - i$, so we can write

$$\frac{z+i}{z-i} = \frac{2i + (z-i)}{z-i} = 1 + \frac{2i}{z-i}$$

for $0 < |z - i| < \infty$.

11. By Cauchy's integral formula, for any z enclosed by Γ_1,

$$f(z) = \frac{1}{2\pi i}\oint_{\Gamma_1}\frac{f(w)}{w-z}\,dw.$$

Because Γ_2 does not enclose z,

$$\frac{1}{2\pi i}\oint_{\Gamma_2}\frac{f(w)}{w-z}\,dw = 0$$

by Cauchy's theorem. The factor of $1/2\pi i$ was included in the last equation so we can add these two integrals to get

$$f(z) = \frac{1}{2\pi i}\left[\oint_{\Gamma_2}\frac{f(w)}{w-z}\,dw + \oint_{\Gamma_1}\frac{f(w)}{w-z}\,dw\right].$$

Orientation on both curves is counterclockwise. In this sum of integrals, L_1 and L_2 are traversed in both directions, so the integrals over these segments are zero. The integrals in square brackets therefore give us the integrals over γ_1 and γ_2, but counterclockwise on γ_1 and clockwise on γ_2. Reversing this orientation on γ_1 so that all integrals are over counterclockwise curves, we have

$$f(z) = \frac{1}{2\pi i}\left[\oint_{\gamma_2}\frac{f(w)}{w-z}\,dw - \oint_{\gamma_1}\frac{f(w)}{w-z}\,dw\right].$$

Now manipulate the $1/(w - z)$ factor in each integral to derive the result we want. For the integral over γ_2, write

$$\begin{aligned}\frac{1}{w-z} &= \frac{1}{w-z_0-(z-z_0)}\\ &= \frac{1}{w-z_0}\frac{1}{1-(z-z_0)/(w-w_0)}\\ &= \frac{1}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n\\ &= \sum_{n=0}^{\infty}\frac{1}{(w-z_0)^{n+1}}(z-z_0)^n.\end{aligned}$$

This geometric expansion is valid because, for w on γ_2,

$$\left|\frac{w-z_0}{z-z_0}\right| < 1.$$

For the integral over γ_1, use the fact that, for w on this curve,

$$\left|\frac{w-z_0}{z-z_0}\right| < 1.$$

Now we have

$$\begin{aligned}\frac{1}{w-z} &= \frac{1}{w-z_0-(z-z_0)}\\ &= \frac{-1}{z-z_0}\frac{1}{1-(w-z_0)/(z-z_0)}\\ &= -\frac{1}{z-z_0}\sum_{n=0}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)\\ &= -\sum_{n=0}^{\infty}(w-z_0)^n\frac{1}{(z-z_0)^{n+1}}.\end{aligned}$$

Substitute these expressions into the sum of integrals representing $f(z)$ and interchange the integrals with the summation to obtain

$$\begin{aligned}f(z) &= \frac{1}{2\pi i}\oint_{\gamma_2}\left(\sum_{n=0}^{\infty}\frac{f(w)}{(w-z_0)^{n+1}}\,dw\right)(z-z_0)^n\\ &\quad+\frac{1}{2\pi i}\oint_{\gamma_1}\left(\sum_{n=0}^{\infty}f(w)(w-z_0)^n\,dw\right)\left(\frac{1}{z-z_0}\right)^{n+1}\\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{\gamma_2}\frac{f(w)}{(w-z_0)^{n+1}}\,dw\right)(z-z_0)^n\\ &\quad+\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{\gamma_1}f(w)(w-z_0)^n\,dw\right)\frac{1}{(z-z_0)^{n+1}}.\end{aligned}$$

Finally, use the deformation theorem to replace these integrals over γ_1 and γ_2 with integrals over Γ, which is any simple closed path in the annulus and enclosing z_0. This gives us

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n,$$

with the integral expressions given for the coefficients c_n.

Chapter 22

Singularities and the Residue Theorem

22.1 Singularities

1. $\cos(z)/z$ has one singularity, a double pole at $z = 0$.

3. $e^{1/z}(z + 2i)$ has an essential singularity at 0.

5. The function has a double pole at 1 and simple poles at i and $-i$.

7. Write
$$\frac{z-i}{z^2+1} = \frac{z-i}{(z+i)(z-i)} = \frac{1}{z+i},$$
so the function has a simple pole at $-i$.

9. The denominator has simple zeros at $1, -1, i, -i$ and these are simple poles of the function because the numerator does not vanish at any of these numbers.

11. $\sec(z) = 1/\cos(z)$ has simple poles at the zeros of $\cos(z)$, which are the simple zeros $(2n + 1)\pi/2$ with n any integer.

13. Suppose f is differentiable at z_0 and $f(z_0) \neq 0$, while g has a pole of order m at z_0. We want to show that the product fg has a pole of order m at z_0.

    Because g has a pole of order m at z_0, the Laurent expansion in some annulus about z_0 has the form
$$g(z) = \frac{k}{(z-z_0)^m} + \sum_{n=-m+1}^{\infty} c_n(z-z_0)^n.$$

with $k \neq 0$. Then

$$(z - z_0)^m g(z) = k + \sum_{n=-m+1}^{\infty} c_n (z - z_0)^{n+m}.$$

If we denote the power series on the right as $h(z)$, then

$$(z - z_0)^m g(z) = h(z),$$

where $h(z_0) = k \neq 0$. Further, in some annulus about z_0,

$$f(z)g(z) = \frac{f(z)h(z)}{(z - z_0)^m}.$$

Because $f(z_0)h(z_0) \neq 0$, $f(z)g(z)$ has a pole of order m at z_0.

22.2 The Residue Theorem

1. The function has simple poles at 1 and $-2i$, both enclosed by γ. Keep in mind that only singularities enclosed by the curve are relevant in evaluating the integral by the residue theorem.

   Compute

$$\begin{aligned}\operatorname{Res}(f, 1) &= \lim_{z \to 1} \frac{d}{dz} \left(\frac{1 + z^2}{z + 2i} \right) \\ &= \lim_{z \to 1} \frac{(z + 2i)(2z) - (1 + z^2)}{(z + 2i)^2} \\ &= \frac{4i}{-3 + 4i},\end{aligned}$$

   and

$$\operatorname{Res}(f, -2i) = \lim_{z \to -2i} \frac{1 + z^2}{(z - 1)^2} = \frac{-3}{-3 + 4i}.$$

   Then

$$\oint_\gamma \frac{1 + z^2}{(z - 1)^2 (z + 2i)} \, dz = 2\pi i \left[\frac{4i}{-3 + 4i} - \frac{3}{-3 + 4i} \right] = 2\pi i.$$

3. The only singularity of e^z/z is a simple pole at 0, and this is not enclosed by γ, so

$$\oint_\gamma \frac{e^z}{z} \, dz = 0$$

   by Cauchy's theorem.

5. The function has simple poles at $\sqrt{6}i$ and $-\sqrt{6}i$, both enclosed by γ. Then

$$\begin{aligned}\oint_\gamma \frac{z+i}{z^2+6}\,dz &= 2\pi i\left[\operatorname{Res}(6,\sqrt{6}i)+\operatorname{Res}(f,-\sqrt{6}i)\right]\\ &= 2\pi i\left[\frac{\sqrt{6}+1}{2\sqrt{6}}+\frac{\sqrt{6}-1}{2\sqrt{6}}\right]=2\pi i.\end{aligned}$$

7. $z/\sinh^2(z)$ has a simple pole at 0 and double poles at $n\pi i$, for every nonzero integer n. The only singularity enclosed by γ is 0, so

$$\oint_\gamma f(z)\,dz = 2\pi i\operatorname{Res}(f,0).$$

Compute this residue as

$$\begin{aligned}\operatorname{Res}(f,0) = \lim_{z\to 0} zf(z) &= \lim_{z\to 0}\left(\frac{z^2}{\sinh^2(z)}\right)\\ &= \lim_{z\to 0}\frac{z^2}{z^2+\frac{1}{6}z^4+\cdots}\\ &= \lim_{z\to 0}\frac{1}{1+\frac{1}{6}z^2+\cdots} = 1.\end{aligned}$$

Then

$$\oint_\gamma \frac{z}{\sinh^2(z)}\,dz = 2\pi i.$$

9. $f(z)$ has simple poles at $i, 3i$ and $-3i$. Only the pole at $-3i$ is enclosed by the curve, so

$$\begin{aligned}&\oint_\gamma \frac{iz}{(z^2+9)(z-i)}\,dz = 2\pi i\operatorname{Res}(f,-3i)\\ &= 2\pi i\lim_{z\to -3i}\frac{iz}{(z-3i)(z-i)} = 2\pi i\left(-\frac{1}{8}\right) = -\frac{\pi i}{4}.\end{aligned}$$

11. $f(z)$ has only one singularity, a simple pole at $-4i$, and this is outside the region bounded by the curve. By Cauchy's theorem,

$$\oint_\gamma \frac{8z-4i+1}{z+4i}\,dz = 0.$$

13. The singularities of $\coth(z) = \cosh(z)/\sinh(h)$ are the zeros of $\sinh(z)$. This means that $\coth(z)$ has simple poles at $n\pi i$, with n any integer. Only the simple pole at 0 is enclosed by the curve, so

$$\oint_\gamma \coth(z)\,dz = \operatorname{Res}(f,0) = 2\pi i\frac{\cosh(0)}{\cosh(0)} = 2\pi i.$$

15. 0 and $4i$ are simple poles of $f(z)$ and both are enclosed by γ, so

$$\begin{aligned}\oint_\gamma \frac{e^{2z}}{z(z-4i)}\,dz &= 2\pi i\,[\mathrm{Res}(f,0)+\mathrm{Res}(f,4i)] \\ &= 2\pi i\left[-\frac{1}{4i}+\frac{e^{4i}}{4i}\right] \\ &= \frac{\pi}{2}[\cos(8)-1+i\sin(8)].\end{aligned}$$

17. z_0 is a zero of order 2 of $h(z)$, but $g(z_0)\neq 0$. We want to show that

$$\mathrm{Res}(g/h,z_0)=\frac{2g'(z_0)}{h''(z_0)}-\frac{2}{3}\frac{g(z_0)h^{(3)}(z_0)}{(h''(z_0))^2}.$$

To do this, first write

$$h(z)=(z-z_0)^2\varphi(z),$$

with $\varphi(z_0)\neq 0$. Then

$$\begin{aligned}\mathrm{Res}(g/h,z_0) &= \lim_{z\to z_0}\frac{d}{dz}\left((z-z_0)^2\frac{g(z)}{h(z)}\right) \\ &= \lim_{z\to z_0}\frac{d}{dz}\left((z-z_0)^2\varphi(z)\frac{g(z)}{(z-z_0)^2\varphi(z)}\right) \\ &= \lim_{z\to z_0}\frac{d}{dz}\left(\frac{g(z)}{\varphi(z)}\right) \\ &= \frac{\varphi(z_0)g'(z_0)-\varphi'(z_0)g(z_0)}{(\varphi(z_0))^2}.\end{aligned}$$

Now,

$$h'(z)=2(z-z_0)\varphi(z)+(z-z_0)^2\varphi'(z).$$

,

$$h''(z)=2\varphi(z)+4(z-z_0)\varphi'(z)+(z-z_0)^2\varphi''(z),$$

and

$$h^{(3)}(z_0)=6\varphi'(z)+6(z-z_0)\varphi''(z)+(z-z_0)^2\varphi^{(3)}(z).$$

Then

$$\varphi(z_0)=\frac{1}{2}h''(z_0)\text{ and }\varphi'(z_0)=\frac{1}{6}h^{(3)}(z_0).$$

Substituting these into the expression for the residue, we have

$$\mathrm{Res}(g/h,z_0)=\frac{2g'(z_0)}{h''(z_0)}-\frac{2}{3}\frac{g(z_0)h^{(3)}(z_0)}{(h''(z_0))^2}.$$

19. By the residue theorem, with $g(z) = z/(2+z^2)$,

$$\begin{aligned}\oint_\gamma \frac{z}{2+z^2}\,dz &= 2\pi i\left[\operatorname{Res}(g,\sqrt{2}i) + \operatorname{Res}(g,-\sqrt{2}i)\right]\\ &= 2\pi i\left[\frac{\sqrt{2}i}{2\sqrt{2}i} + \frac{-\sqrt{2}i}{-2\sqrt{2}i}\right]\\ &= 2\pi i.\end{aligned}$$

To use the argument principle, write

$$g(z) = \frac{f'(z)}{f(z)} = \frac{1}{2}\frac{2z}{2+z^2},$$

with $f(z) = 2+z^2$. Then $f'/f = 2g$. Now $f(z)$ has two simple zeros enclosed by γ, and no poles, so $Z = 2$, $P = 0$, and

$$\begin{aligned}\oint_\gamma \frac{z}{2+z^2}\,dz &= \frac{1}{2}\oint_\gamma \frac{2z}{1+z^2}\,dz\\ &= \frac{1}{2}(2\pi i)(Z-P) = 2\pi i.\end{aligned}$$

21. $g(z) = (z+1)/(z^2+2z+4)$ has simple poles at $-1\pm\sqrt{3}i$ enclosed by γ. By the residue theorem

$$\begin{aligned}\oint_\gamma \frac{z+1}{z^2+2z+4}\,dz &= 2\pi i\left[\operatorname{Res}(g,-1-\sqrt{3}) + \operatorname{Res}(g,-1+\sqrt{3})\right]\\ &= 2\pi i\left[\frac{1-1-\sqrt{3}i}{2(-1-\sqrt{3}i)+2} + \frac{-1+\sqrt{3}i+1}{2(-1+\sqrt{3}i)+2}\right]\\ &= 2\pi i\left(\frac{1}{2}+\frac{1}{2}\right) = 2\pi i.\end{aligned}$$

To use the argument principle, write

$$\frac{z+1}{z^2+2z+4} = \frac{1}{2}\frac{f'(z)}{f(z)}$$

where $f(z) = z^2+2z+4$. $f(z)$ has $z = 2$ zeros enclosed by γ and no poles ($P = 0$), so

$$\oint_\gamma \frac{2z+2}{z^2+2z+4}\,dz = \pi i(Z-P) = 2\pi i.$$

22.3 Evaluation of Real Integrals

1. With $z = e^{i\theta}$,

$$\cos(\theta) = \frac{1}{2}\left(z+\frac{1}{z}\right) \text{ and } d\theta = \frac{1}{iz}\,dz$$

so

$$\int_0^{2\pi} \frac{1}{2-\cos(\theta)}\,d\theta = \oint_\gamma \frac{1}{2-\frac{1}{2}(z+1/z)}\frac{1}{iz}\,dz$$
$$= 2i\oint_\gamma \frac{1}{z^2-4z+1}\,dz.$$

The integrand has simple poles at $z_1 = 2-\sqrt{3}$ and $z_2 = 2+\sqrt{3}$. Only z_1 is enclosed by γ, and

$$\operatorname{Res}(f, 2-\sqrt{3}) = \frac{1}{2(2-\sqrt{3})-4} = -\frac{1}{2\sqrt{3}}.$$

Then

$$\int_0^{2\pi} \frac{1}{2-\cos(\theta)}\,d\theta = 2i(2\pi i)\frac{-1}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}.$$

3. $f(z) = 1/(1+z^6)$ has simple poles in the upper half-plane at $z_1 = i$, $z_2 = (\sqrt{3}+i)/2$, and $z_3 = (-\sqrt{3}+i)/2$. The residues of $f(z)$ at these poles are

$$\operatorname{Res}(f, z_j) = \frac{1}{6z_j^5} = -\frac{1}{6}z_j,$$

so

$$\int_{-\infty}^{\infty} \frac{1}{1+x^6}\,dx = 2\pi i\left[\frac{1}{6}(z_1+z_2+z_3)\right] = \frac{2\pi}{3}.$$

Then

$$\int_0^{\infty} \frac{1}{1+x^6}\,dx = \frac{\pi}{3}.$$

5. Let

$$f(z) = \frac{ze^{2iz}}{z^4+16}.$$

f has simple poles in the upper half-plane at $z_1 = (1+i)/\sqrt{2}$ and $z_2 = (-1+i)/\sqrt{2}$. Compute the residues:

$$\operatorname{Res}(f, z_1) = \frac{e^{2\sqrt{2}i(-1+i)}}{16i} \text{ and } \operatorname{Res}(f, z_2) = \frac{e^{2\sqrt{2}}(-1-i)}{-16i}$$

to obtain

$$\int_{-\infty}^{\infty} \frac{x\sin(2x)}{x^4+16}\,dx = \operatorname{Im}\left[2\pi i\left(\frac{e^{-2\sqrt{2}}}{8}\right)\left(\frac{e^{2\sqrt{2}i}-e^{-2\sqrt{2}i}}{2i}\right)\right]$$
$$= \frac{\pi e^{-2\sqrt{2}}}{4}\sin(2\sqrt{2}).$$

7. First use the identity

$$\cos^2(x) = \frac{1}{2}(1+\cos(2x))$$

to write

$$\int_{-\infty}^{\infty} \frac{\cos^2(x)}{(x^2+4)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1+\cos(2x)}{(x^2+4)^2}\,dx.$$

Let

$$f(z) = \frac{1+e^{2iz}}{(z^2+4)^2}.$$

Then f has a pole of order 2 in the upper half-plane at $2i$. Compute

$$\text{Res}(f,2i) = \lim_{z\to 2i} \frac{d}{dx}\left[\frac{1+e^{2iz}}{(z+2i)^2}\right] = \frac{1+5e^{-1}}{32i}.$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\cos^2(x)}{(x^2+4)^2}\,dx &= \frac{1}{2}\text{Re}\left[2\pi i\left(\frac{1+5e^{-1}}{32i}\right)\right]\\ &= \frac{\pi}{32}(1+5e^{-4}).\end{aligned}$$

9. Let $f(z) = z^2/(z^2+4)^2$. The only singularity of f in the upper half-plane is $2i$, which is a double pole. Compute

$$\text{Res}(f,2i) = \lim_{z\to 2i} \frac{d}{dz}\left[\frac{z^2}{(z+2i)^2}\right] = -\frac{i}{8}.$$

Then

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)^2}\,dx = 2\pi i\left(-\frac{i}{8}\right) = \frac{\pi}{4}.$$

11. Let

$$f(z) = \frac{e^{i\alpha z}}{z^2+1}.$$

The only singularity f has in the upper half-plane is a simple pole at i. Compute

$$\text{Res}(f,i) = \frac{e^{-\alpha}}{2i}.$$

Then

$$\int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{x^2+1}\,dx = 2\pi i\left(\frac{e^{-\alpha}}{2i}\right) = \pi e^{-\alpha}.$$

13. Begin with

$$\begin{aligned}
&\int_0^{2\pi} \frac{1}{\alpha^2\cos^2(\theta)+\beta^2\sin^2(\theta)}\,d\theta \\
&= \oint_\gamma \frac{1}{\alpha^2(z+1/z)^2/4-\beta^2(z-1/z)^2/4}\frac{1}{iz}\,dz \\
&= \frac{4}{i}\oint_\gamma \frac{z}{(\alpha^2-\beta^2)z^4+2(\alpha^2+\beta^2)z^2+(\alpha^2-\beta^2)}\,dz.
\end{aligned}$$

Singularities of the integrand satisfy

$$z^2 = \frac{\beta-\alpha}{\beta+\alpha} \text{ or } z^2 = \frac{\beta-\alpha}{\beta+\alpha}.$$

Because α and β are positive,

$$\left|\frac{\beta-\alpha}{\beta+\alpha}\right| < 1 \text{ and } \left|\frac{\beta+\alpha}{\beta-\alpha}\right| > 1.$$

The simple poles enclosed by the unit circle are the square roots z_1 and z_2 of $(\beta-\alpha)/(\beta+\alpha)$. The residue of the integrand at each of these poles can be computed using Corollary 22.1. Omitting the arithmetic, we obtain

$$\mathrm{Res}(f, z_j) = \frac{1}{8\alpha\beta}$$

for $j = 1, 2$. Then

$$\int_0^{2\pi} \frac{1}{\alpha^2\cos^2(\theta)+\beta^2\sin^2(\theta)}\,d\theta = \frac{4}{i}(2\pi i)\frac{2}{8\alpha\beta} = \frac{2\pi}{\alpha\beta}.$$

15. Let Γ be the given rectangular path. The four sides are:

$$\begin{aligned}
&\Gamma_1 : z = t, -R \le t \le R, \\
&\Gamma_2 : z = R + it, 0 \le t \le \beta, \\
&\Gamma_3 : z = t + i\beta, -R \le t \le R, \\
&\Gamma_4 : z = -R + it, 0 \le t \le \beta.
\end{aligned}$$

These are, respectively, the lower side, right side, top side and left side of the rectangle. In carrying out the integrations, limits of integration must be consistent with counterclockwise orientation of Γ.

Because e^{-z^2} is differentiable for all z, then by Cauchy's theorem,

$$\oint_\Gamma e^{-z^2}\,dz = 0 = \sum_{j=1}^{4}\int_{\Gamma_j} e^{-z^2}\,dz.$$

Evaluate each of the four integrals in the sum on the right as follows.

$$\int_{\Gamma_1} e^{-z^2}\,dz = \int_{-R}^{R} e^{-t^2}\,dt,$$

$$\begin{aligned}\int_{\Gamma_2} e^{-z^2}\,dz &= \int_0^\beta e^{-(r^2+2Rti-t^2)} i\,dt \\ &= ie^{-R^2}\int_0^\beta e^{t^2}[\cos(2Rt) - i\sin(2Rt)]\,dt,\end{aligned}$$

$$\begin{aligned}\int_{\Gamma_3} e^{-z^2}\,dz &= \int_{-R}^{R} e^{-(t^2+2\beta ti-\beta^2)}\,dt \\ &= e^{-\beta^2}\int_{-R}^{R} e^{-t^2}[\cos(2\beta t) - i\sin(2\beta t)]\,dt,\end{aligned}$$

and

$$\begin{aligned}\int_{\Gamma_4} e^{-z^2}\,dz &= \int_\beta^0 e^{-(R^2-2Rti-t^2)} i\,dt \\ &= ie^{-R^2}\int_0^\beta [e^{t^2}[-\cos(2\beta t) - i\sin(2\beta t)]\,dt.\end{aligned}$$

Now let $R \to \infty$. The terms having a factor of e^{-R^2} go to zero in the limit, and upon adding these integrals over the sides of the rectangle, we obtain, using x as the variable of integration on the line,

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx - e^{-\beta^2}\int_{-\infty}^{\infty}[\cos(2\beta x) - i\sin(2\beta x)]\,dx = 0.$$

Now, $e^{-x^2}\sin(2\beta x)$ is an odd function on the real line, so

$$\int_{-\infty}^{\infty} e^{-x^2}\sin(2\beta x)\,dx = 0.$$

We are therefore lift with

$$e^{\beta^2}\int_{-\infty}^{\infty}\cos(2\beta x)\,dx = \int_{-\infty}^{\infty} e^{-x^2}\,dx.$$

Finally, use the known result that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

to conclude that

$$\int_{-\infty}^{\infty} e^{-x^2}\cos(2\beta x)\,dx = \sqrt{\pi}e^{-\beta^2}.$$

Finally, because $e^{-x^2}\cos(2\beta x)$ is an even function on the real line, then

$$\int_0^\infty e^{-x^2}\cos(2\beta x)\,dx = \frac{\sqrt{\pi}}{2}e^{-\beta^2}.$$

17. First observe that, because the integrand is an even function,

$$\int_0^\infty \frac{x\sin(\alpha x)}{x^4+\beta^4}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x\sin(\alpha x)}{x^4+\beta^4}\,dx.$$

Now,

$$f(z) = \frac{ze^{i\alpha z}}{z^4+\beta^4}$$

has simple poles in the upper half-plane at $z_1 = \beta e^{i\pi/4}$ and $z_2 = \beta e^{3\pi i/4}$. Compute the residues of f at these poles:

$$\mathrm{Res}(f, z_k) = \left[\frac{ze^{i\alpha z}}{4z^3}\right]_{z=z_k} = \frac{e^{i\alpha\beta z_k}}{4z_k^2}.$$

In particular,

$$\mathrm{Res}(f, z_1) = \frac{1}{4\beta^2 i}e^{i\alpha\beta e^{i\pi/4}} \text{ and } \mathrm{Res}(f, z_2) = \frac{1}{-4\beta^2 i}e^{i\alpha\beta e^{3i\pi/4}}.$$

Then

$$\begin{aligned}\int_0^\infty \frac{x\sin(\alpha x)}{x^4+\beta^4}\,dx &= \frac{1}{2}\mathrm{Im}\left[\frac{2\pi i}{4\beta^2}\left(e^{i\alpha\beta(1+i)/\sqrt{2}} - e^{i\alpha\beta(-1+i)/\sqrt{2}}\right)\frac{1}{i}\right] \\ &= \frac{\pi e^{-\alpha\beta/\sqrt{2}}}{2\beta^2}\sin\left(\frac{\alpha\beta}{\sqrt{2}}\right).\end{aligned}$$

Chapter 23

Conformal Mappings and Applications

23.1 Conformal Mappings

For Problems 1–3, the image of the given rectangle is given as a graph for each part of the problem.

1. (a) The rectangle defined by $0 \leq x \leq \pi, 0 \leq y \leq \pi$ maps to the sector

$$1 \leq r \leq e^{\pi}, 0 \leq \theta \leq \pi.$$

See Figure 23.1.

(b) This rectangle maps to the sector

$$\frac{1}{e} \leq r \leq e, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

See Figure 23.2.

(c) The rectangle maps to the sector

$$1 \leq r \leq e, 0 \leq \theta \leq \frac{\pi}{4}.$$

See Figure 23.3.

(d) The rectangle maps to the sector

$$e \leq r \leq e^2, 0 \leq \theta \leq \pi.$$

See Figure 23.4.

(e) The rectangle maps to the sector

$$\frac{1}{e} \leq r \leq e^2, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

See Figure 23.5.

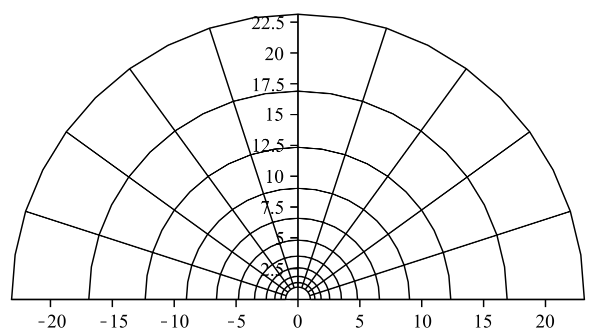

Figure 23.1: Image of the rectangle $0 \leq x \leq \pi, 0 \leq y \leq \pi$ under $w = e^z$.

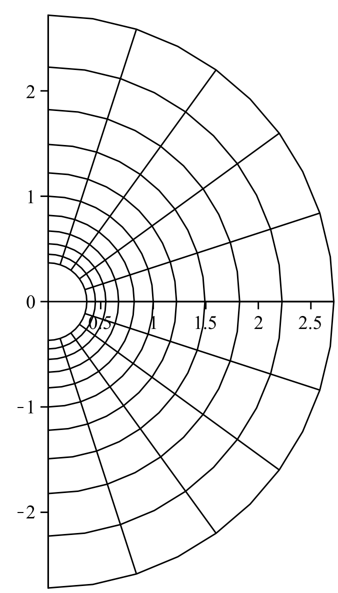

Figure 23.2: Image of $-1 \leq x \leq 1, -\pi/2 \leq y \leq \pi/2$.

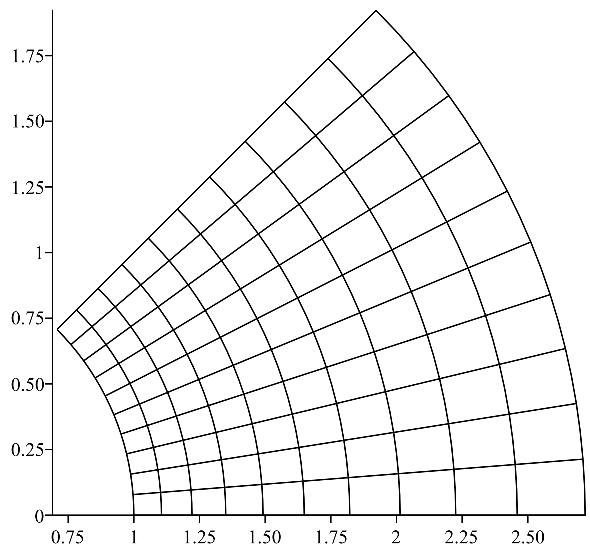

Figure 23.3: Image of the rectangle $0 \leq x \leq 1, 0 \leq y \leq \pi/4$.

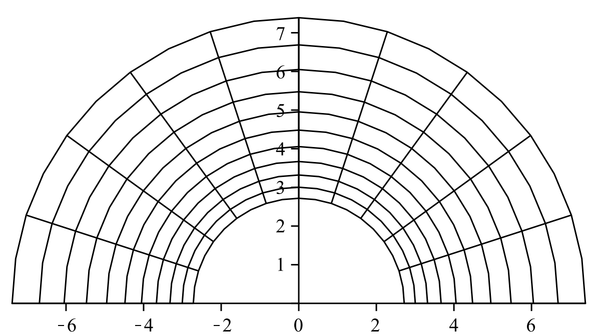

Figure 23.4: Image of the rectangle $1 \leq x \leq 2, 0 \leq y \leq \pi$.

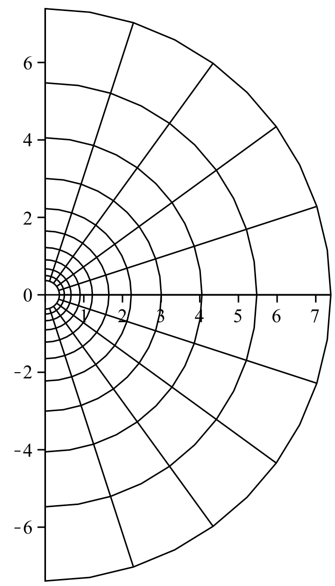

Figure 23.5: Image of the rectangle $-1 \le x \le 2, -\pi/2 \le y \le \pi/2$.

3. For the images of the given rectangles of parts (a) through (e), see Figures 23.6–23.10, respectively.

5. The analysis proceeds like that of Problem 4. Let $z = re^{i\theta}$. If $\pi/6 \le \theta \le \pi/3$, then $\pi/2 \le 3\theta \le \pi$, so image points under this mapping lie in the second quadrant. It is routine to verify that this mapping is onto the second quadrant.

7. Using some of the analysis from Problem 6, a half-line $\theta = k$ maps to points $u + iv$ with

$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos(k), v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin(k).$$

If $\sin(k) \neq 0$ and $\cos(k) \neq 0$, then a little algebraic manipulation shows that

$$\frac{u^2}{\cos^2(k)} - \frac{v^2}{\sin^2(k)} = 1.$$

This is the equation of a hyperbola with foci $(\pm c, 0)$, where

$$c^2 = \cos^2(k) + \sin(k) = 1.$$

Finally, the cases $\cos(k) = 0$ and $\sin(k) = 0$ must be considered separately. If $\cos(k) = 0$, then $k = (2n+1)\pi/2$. Now $u = 0$ and $-\infty < v < \infty$, so the image is the imaginary axis in the $x-$plane.

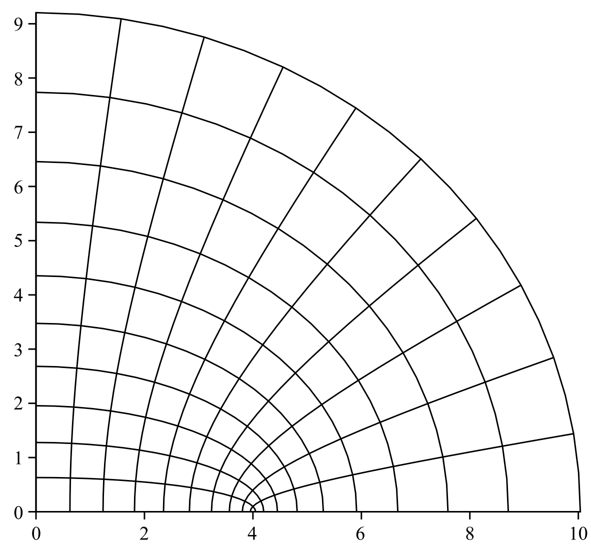

Figure 23.6: Image of the rectangle $0 \leq x \leq \pi/2, 0 \leq y \leq \pi/2$.

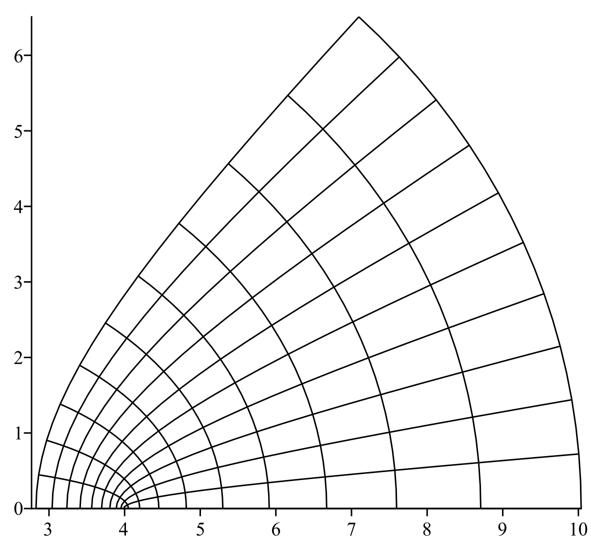

Figure 23.7: Image of the rectangle $\pi/4 \leq x \leq \pi, 0 \leq y \leq \pi/2$.

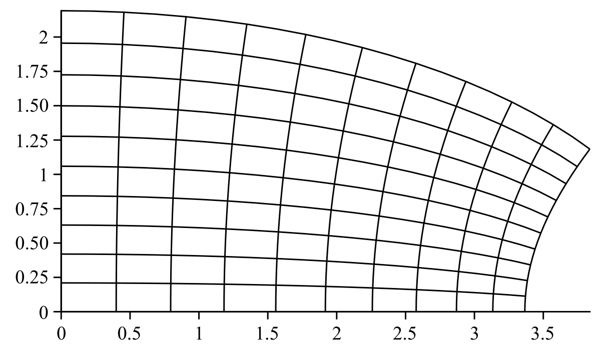

Figure 23.8: Image of the rectangle $0 \leq x \leq 1, 0 \leq y \leq \pi/6$.

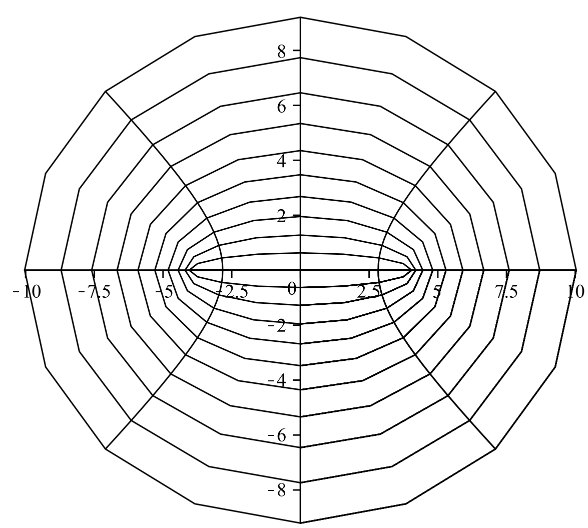

Figure 23.9: Image of the rectangle $\pi/2 \leq x \leq 3\pi, 0 \leq y \leq \pi/2$.

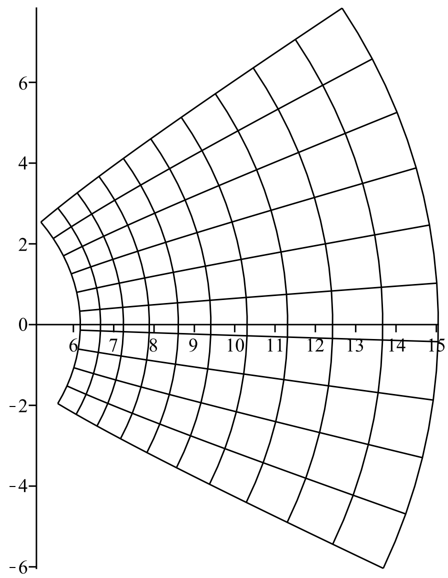

Figure 23.10: Image of the rectangle $1 \leq x \leq 2, 1 \leq y \leq 2$.

If $\sin(k) = 0$, then $k = n\pi$ with n an integer. Now $v = 0$ and

$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)(-1)^n.$$

This is the half-interval $u \geq 1$ on the real axis in the $w-$plane if n is even, and $u \leq -1$ if n is odd.

9. Write

$$w = 2z^2 = 2(x + iy)^2 = 2(x^2 - y^2) + 4ixy = u + iv.$$

Then vertical line $x = 0$ maps to $u = -2y^2, v = 0$, so the image is the negative $u-$ axis. Other vertical lines $x = a \neq 0$ map onto parabolas

$$u = 2a^2 - \frac{v^2}{2a^2}.$$

The horizontal line $y = 0$ maps to $u = 2y^2 \geq 0$, the positive $u-$axis. Other horizontal lines $y = b \neq 0$ map onto the parabolas

$$u = \frac{v^2}{8b^2} - 2b^2.$$

Figure 23.11 shows the image of the rectangle $0 \leq x \leq 3/2$, $-3/2 \leq y \leq 3/2$.

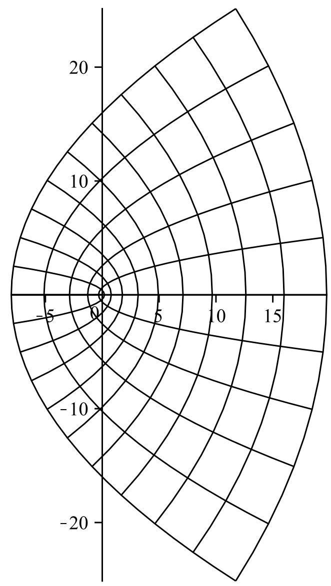

Figure 23.11: Image of the rectangle in Problem 9 with $\alpha = 2$.

11. If $\text{Re}(z) = -4$, then $(z + \overline{z})/2 = -4$, so $z + \overline{z} = -8$. Now, $w = 2i/z$, so $z = 2i/w$, so

$$z + \overline{z} = \frac{2i}{w} - \frac{2i}{\overline{w}} = -8.$$

Multiply this by $w\overline{w}$ and rearrange terms to obtain

$$8w\overline{w} - 2i(w - \overline{w}) = 0.$$

Now put $w = u + iv$ to get

$$2(u^2 + v^2) + v = 0,$$

or

$$u^2 + \left(v + \frac{1}{4}\right)^2 = \frac{1}{4}.$$

This is the equation of a circle of radius $1/2$ centered at $(0, -1/4)$ in the $w-$ plane. This is the image of the line $x = -4$ under the given mapping.

13. From the mapping, solve for z:

$$z = \frac{-1}{w + i}.$$

Substitute this into the given line to obtain

$$\frac{1}{2}\left(\frac{-1}{w+i} - \frac{1}{\overline{w} - i}\right) + \frac{1}{2i}\left(\frac{-1}{w+i} + \frac{1}{\overline{w} - i}\right) = 1.$$

Multiply this equation by $2i(w+i)(\overline{w}-i)$ and rearrange terms, putting $w = u + iv$ to obtain

$$4(u^2+v^2)+7v+u=3.$$

Complete the square to write this as

$$\left(u+\frac{1}{8}\right)^2+\left(v+\frac{7}{8}\right)^2=\frac{1}{32}.$$

This is a circle of radius $1/2\sqrt{2}$ and center $(-1/8, -7/8)$, and is the image of the given line.

15. Invert the mapping to obtain

$$z=\frac{5+iw}{2-w}.$$

Then

$$\begin{aligned}
z-\overline{z}=2\mathrm{Re}(z)=2\mathrm{Re}\left(\frac{5-v+iu}{2-u-iv}\right)\\
=\frac{2((5-v)(2-u)-uv)}{(u-2)^2+v^2}\\
=\frac{20-4v-10u}{(u-2)^2+v^2}.
\end{aligned}$$

Next,

$$\frac{1}{2i}(z-\overline{z})=\mathrm{Im}(z)=\frac{(2-u)u+(5-v)v}{(u-2)^2+v^2}.$$

Substitute these into the equation of the given line and clear fractions to obtain

$$(u-1)^2+\left(v+\frac{19}{4}\right)^2=\frac{377}{16}.$$

This is the equation of a circle with radius $\sqrt{377}/4$ and having center $(1, -19/4)$.

17. Substitute the given values into equation (23.1) to obtain

$$(1-w)(1+2i)(-1)(3-z)=(1-z)(1)(1+i)(1-i-w).$$

Solve for w to obtain

$$w=\frac{(1+4i)z-(3+8i)}{(2+3i)z-(4+7i)}.$$

19. Here $w_3 = \infty$ so use equation (23.2), obtaining

$$(1 + i - w)(1 - 2i)(4 - z) = (1 - z)(-2 + 2i)(4 - 2i).$$

Then

$$w = \frac{(33 + i)z - (48 + 16i)}{5(z - 4)}.$$

21. Using equation (23.1), we find that

$$w = \frac{(3 + 22i) + 4 - 75i}{(2 + 3i) - (21 - 4i)}.$$

23. If we require that a conformal mapping be differentiable, then immediately $T(z) = \overline{z}$ is disqualified. But it is also easy to see directly that this mapping reverses orientation. For example, let C_1 be the nonnegative real axis and C_2 the nonnegative imaginary axis. The sense of rotation from C_1 to C_2 is counterclockwise. But T maps C_1 to itself and C_2 to the negative imaginary axis, reversing the orientation to clockwise. Therefore T is not conformal.

25. Let

$$T(z) = \frac{az + b}{cz + d}.$$

By the argument of Problem 24, if T is not a translation or the identity mapping, then T can have at most two fixed points. Therefore, if T has three fixed points, then T is either a translation or the identity map. But a translation has no fixed point, so in this case T must be the identity map.

27. Given z_2, z_3, z_4, let P be the unique bilinear transformation that maps

$$z_2 \to 1, z_3 \to 0, z_4 \to \infty.$$

Then by definition of the cross ratio,

$$P(z_1) = [z_1, z_2, z_3, z_4].$$

Now let T be any bilinear transformation. Then

$$[T(z_1), T(z_2), T(z_3), T(z_4)] = R(T(z_1)),$$

where R is the unique bilinear transformation that maps

$$T(z_2) = 1, T(z_3) = 0, T(z_4) = \infty.$$

Then $R \circ T = P$. Then

$$\begin{aligned}[T(z_1), T(z_2), T(z_3), T(z_4)] &= R(T(z_1)) \\ &= P(z_1) = [z_1, z_2, z_3, z_4].\end{aligned}$$

29. In the definition of cross ratio, w_2, w_3 and w_4 all lie on an (extended) line, the real axis. Because bilinear transformations map lines/circles to lines/circles, then $[z_1, z_2, z_3, z_4]$ is real exactly when $z_1, z_2, z_3, z_4]$ all lie on the same line or circle.

23.2 Construction of Conformal Mappings

If a conformal mapping is requested between two domains, there will in general be many different possible mappings. In each of the solutions below, one mapping is found, but other approaches may yield other suitable mappings.

1. Both domains are open disks, having radii 3 and 6, and different centers, 0 and $1-i$, respectively. We can map $|z|<3$ ont $|w-1+i|<6$ by using a scaling factor of 2 and a translation to superimpose the center of the initial domain onto the center of the image domain. Thus compose

$$z \to 2z \to 2z+1-i.$$

One mapping that does what we want is

$$w = 2z+1-i.$$

Now

$$|w-1+i| = 2|z| = 2(3) = 6$$

if $|z|=3$.

3. We will need an inversion at some stage because we are mapping the interior of a disk to the exterior of another disk. First translate by $w_1 = z+2i$, so the image disk in the $w-$plane has the origin as its center. Next invert by

$$w_2 = \frac{1}{z+2i}.$$

Next scale by a factor of 2 to match the radii of the bounding circles:

$$w_3 = 2w_2 = \frac{2}{z+2i}.$$

Finally, translate to have center 2:

$$w_4 = w_3 + 3 = \frac{2}{z+2i} + 3 = \frac{3z+2+6i}{z+2i}.$$

5. We can map the line $\text{Re}(z)=0$ to the circle by $|w|=4$ by a bilinear transformation. The domain $\text{Re}(z)<0$ consists of numbers to the left of the imaginary axis, which is the boundary of this domain. Choose three points on this axis, ordered upward so the region $\text{Re}(z)<0$ is on the left as we walk up the line. Next choose three points on the image circle $|w|=4$, counterclockwise so the interior of the circle is on the left as we walk counterclockwise around the circle. Convenient choices are

$$z_1 = -i, z_2 = 0, z_3 = i \text{ and } w_1 = -4i, w_2 = 4, w_3 = -4i.$$

Using equation (23.1), we find the bilinear transformation mapping $z_j \to w_j$:

$$w = T(z) = 4\left(\frac{1+z}{1-z}\right).$$

As a check, $z = -1$, which has negative real part, maps to 0, interior to the disk $|w| < 4$. Thus T maps $\mathrm{Re}(z) < 1$ to $|w| < 4$, rather than to the exterior $|w| > 4$.

7. Because the boundary of the wedge in the $w-$plane is not a circle or line, a bilinear transformation will not work here. However, wedges suggest polar representations. Let $z = re^{i\theta}$ for $0 < \theta < \pi$. These are points in the upper half-plane. Let

$$w = z^{1/3} = r^{1/3}e^{i\theta/3} = \rho e^{i\varphi}.$$

Here $\rho > 0$ and $0 \le \varphi \le \pi/3$. This mapping is conformal because

$$\frac{dw}{dz} = \frac{1}{3}z^{-2/3} \neq 0$$

for z in the upper half-plane, and the mapping takes the open upper half-plane onto the open wedge $0 < \theta < \pi/2$.

9. The solution of this problem requires some familiarity with the gamma and beta functions.

To show that f maps the upper half-plane onto the given rectangle, we will evaluate the function at $-1, 0, 1$ and ∞ and then show that these are the vertices of that rectangle.

First, it is obvious that $f(0) = 0$. Next,

$$\begin{aligned} f(1) &= 2i\int_0^1 (\xi^2-1)^{-1/2}\xi^{-1/2}\,d\xi \\ &= 2i\int_0^1 \frac{(1-\xi^2)^{-1/2}}{i}\xi^{-1/2}\,d\xi \\ &= 2\int_0^1 (1-\xi^2)^{-1/2}\xi^{-1/2}\,d\xi. \end{aligned}$$

Let $\xi = u^{1/2}$ to obtain

$$\begin{aligned} f(1) &= \int_0^1 (1-u)^{-1/2}u^{-3/4}\,du \\ &= B(1/4, 1/2) = \frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)} = c, \end{aligned}$$

in which $B(x, y)$ is the beta function and $\Gamma(x)$ is the gamma function. Next, write

$$f(-1) = 2i\int_0^1 (\xi^2-1)^{-1/2}\xi^{-1/2}\,d\xi.$$

Let $\xi = -u$ to obtain

$$f(-1) = 2i \int_0^1 (1-u^2)^{-1/2} u^{-1/2}\, du$$
$$= iB(1/4, 1/2) = i\frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)} = ic.$$

Finally,

$$f(\infty) = 2i \int_0^\infty (\xi+1)^{-1/2}(\xi-1)^{-1/2}\xi^{-1/2}\, d\xi$$
$$= 2i \int_0^1 (\xi+1)^{-1/2}(\xi-1)^{-1/2}\xi^{-1/2}\, d\xi$$
$$+ 2i \int_1^\infty (\xi+1)^{-1/2}(\xi-1)^{-1/2}\xi^{-1/2}\, d\xi.$$

The first integral in the last line of the last equation is $B(1/4, 1/2)$. In the second integral, put $\xi = 1/u$ to get

$$f(\infty) = c + 2i \int_1^0 \left(\frac{1+u}{u}\right)^{-1/2} \left(\frac{1-u}{u}\right)^{-1/2} u^{1/2} \left(\frac{1}{u^2}\right) du$$
$$= c + 2i \int_0^1 (1-u^2)^{-1/2} u^{-1/2}\, du = (1+i)u.$$